ELSEVIER

Physica B 296 (2001) 280–288

PHYSICA B

www.elsevier.com/locate/physb

Scaling of the conductance in anisotropic 2d systems

Xiaosha Wang[a], Qiming Li[a], C.M. Soukoulis[a,b,*]

[a]Ames laboratory-USDOE and Department of Physics and Astronomy, Iowa State University, Ames, IA 50011, USA
[b]Research Center of Crete and Department of Physics, University of Crete, P.O. Box 1527, 71110 Heraklion, Crete, Greece

Abstract

We investigate the scaling of conductances in anisotropic two-dimensional systems. The conductance and its distribution in the two directions are found to be approximately the same, once the dimension of the system in each direction is chosen to be proportional to the localization length. At the localization–delocalization transition under a strong magnetic field, the geometrical mean of the conductance in the two directions is not universal. The distribution of conductances at the critical point also show distinctive features in the two directions. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Conductance; Localization; Anisotropic systems; Metal-to-insulator transition

1. Introduction

Localization in anisotropic systems attracted much attention recently [1–9]. The renewed interest is partly spurred by the attempt to understand the normal state transport properties in different directions of the high-T_c cuprate superconductors which is a layered material [10]. Although it is generally believed that anisotropy in the form of anisotropic hopping does not change the universality and the critical behavior of the problem, the exact form of the scaling function is expected to depend on anisotropy in the form of anisotropic physical parameters such as anisotropic hopping integrals or geometrical aspect ratios [11–13]. These details of the localization behavior in such systems have never been examined until recently. In an extensive numerical calculation [2] in three-dimensional systems with the transfer matrix method coupled with finite size scaling technique, it is clearly demonstrated that the states become localized at the same critical disorder in all the directions in highly anisotropic systems. The scenario that states become localized in one direction and extended in other cannot occur. However, it was also pointed out in that work [2] that the large disparity between the values of the coherent lengths in different directions makes the transport property of such an anisotropic system much more complex at finite temperatures. The scaling property in such anisotropic systems was also examined recently [14]. It was proposed that the scaling function should solely depend on the conductances in the different directions, a natural extension of the original one-parameter scaling theory which was valid only in isotropic systems (For a recent review see [15]). This hypothesis was successfully applied

* Correspondence address: Iowa State University, Ames Laboratory-USDOE, Dept. of Physics and Astronomy, Ames, IA 50011, USA. Fax: + 1-515-294-0689.

E-mail address: soukoulis@ameslab.gov (C.M. Soukoulis).

PII: S0921-4526(00)00811-5

to the scaling of the finite size localization lengths in two-dimensional anisotropic systems. Numerical investigation on scaling of conductance is much more difficult.

For conductance, a simple application of the scaling idea can be made for a special aspect ratio. When the dimension of the system size is chosen to be directly proportional to the localization length, the system should behave effectively isotropic. The scaling function of such a special system should then be the same as that of the isotropic system. This, in fact, has been proposed previously [16–19] but has never been checked before until the recent work. The difficulty in implementing such a procedure lies in the fact that the localization lengths are usually not known a priori. Nevertheless, it was found through numerical calculations [14] that this can indeed be done. With an anisotropy of hopping integral 1:10, the conductances of two-dimensional rectangular samples of different sizes but the same aspect ratio 10:1 were found to be the same in the two directions for conductances in a range of ten orders of magnitude. A particular question raised was that would the distribution of conductances in the two directions be also identical, as expected from the generalization of scaling ideas to conductance distributions [20,21].

The other important question concerns about the universality of the critical conductance at the critical point of the Anderson transition [22,23]. From the generalized scaling functions, it can be established [14] that the geometrical mean of the localization length ratio $\Lambda_c \equiv \lambda_M/M$ is a constant independent of the anisotropy. Numerical calculations in both two- [14] and three-dimensional [1] systems strongly support such a claim. But the same cannot be said on the conductances. In fact, numerical results [1] in three-dimensional anisotropic systems do not support a universal conductance for the geometrical mean. In two-dimensional systems, a localization–delocalization transition occurs only when time reversal symmetry is broken. A well-known example is the integer quantum Hall plateau transition occurring in two-dimensional gas under a strong magnetic field (For reviews see [24,25]). A recent calculation [26] based on the network model [27] found that the geometrical mean of the critical conductances is universal within the uncertainty of the data at the critical points with anisotropic coupling. The universal value, $\langle G_c \rangle = 0.58e^2/h$, is also different from the value predicted by analytical theory [28], $\sigma^c_{xx} = \frac{1}{2}(e^2/h)$. However, our calculation with the tight-binding model obtained, $\langle G_c \rangle = 0.506(e^2/h)$ in isotropic systems. This small but significant difference is in odds with the fundamental universality hypothesis [22,23] which states that the microscopic details are not relevant within a class of models characterized by a few fundamental symmetries. It is therefore critical to investigate if in the two-dimensional anisotropic systems, there is a universality in the critical conductance. These issues will be the central theme of the paper.

2. Formalism

We consider the following tight-binding Hamiltonian for an anisotropic 2d disordered model under a strong magnetic field:

$$H = \sum_n \varepsilon_n |n\rangle\langle n| + \sum_{\langle nm \rangle} (v_{nm}|n\rangle\langle m| + \text{c.c.}), \tag{1}$$

where n and m label the sites of a square lattice. The on-site energies ε_n are independently distributed at random within the interval $[-W/2, W/2]$. The second term is taken over all pairs of nearest-neighbor sites. The complex hopping integral v_{nm} is anisotropic and carries the phase due to applied magnetic fields via the standard Peierls substitution,

$$v_{nm} = t_{nm}\, e^{-(i2\pi e/hc)\int_n^m \vec{A}\, d\vec{l}}. \tag{2}$$

The hopping integral $t_{nm} = 1$ or $t\ (<1)$, depending on hopping directions. As a convention, we have assigned the direction with the large ($t_{nm} = 1$) and the small ($t_{nm} = t$) hopping value as the parallel (||) and the perpendicular ($\perp$) directions, respectively. Periodic boundary conditions are applied in the transverse direction.

The zero temperature two terminal conductance in a rectangular sample of width M and length L is given by the following multichannel Landauer formula [29–31]:

$$G = \frac{e^2}{h} Tr(T^\dagger T), \tag{3}$$

where T is the total transmission matrix through the disordered sample. We calculate T with the standard transfer matrix method. Keep in mind that G defined here is for one spin only.

3. Results and discussion

Since in the localization problem the relevant length scales are the localization lengths, the conductance should become isotropic if the dimensions of the sample are chosen to be proportional to the localization length [12,13]. This means, for anisotropic systems, the shape of the sample should be rectangular with the weak coupling direction as the short dimension, in order for the system to have isotropic conductances in the two directions. Once this is done at some length scale, the isotropic property is expected to maintain when the system size is scaled up with the same factor, according to the one parameter scaling theory [16–19]. This was tested [14] in a 2d system with $t = 0.1$ at $E = 0$ and $W = 3.6$. The aspect ratio was chosen to be 1:10, approximately the localization length ratio. In Fig. 1, we reproduce the geometrical averaged conductance in the two directions when the system is scaled up by up to 4 times. It clearly shows the conductance remains approximately isotropic even though the conductance has decreased by 8 orders of magnitude. Since we do not know exactly the localization length in the two directions, apparently the aspect ratio, 10:1 we chose is only approximately equal to the localization length at $E = 0$ and $W = 3.6$. As a result, the conductances in the two directions are not exactly equal. However, the difference is small compared with the 8 orders of magnitude changes in the value of the conductances. In Fig. 2, we show the statistical distributions of the logarithmic of the conductances in the two directions at different sizes. The distribution is clearly Gaussian, indicating that the state is localized. This is also the reason why a geometric mean is chosen in Fig. 1 because the arithmetic mean is known to be ill-behaved for localized states. The distributions in the two directions do not show any significant difference, indicating that the system indeed cannot be differentiated from an isotropic one, in agreement with the one-parameter scaling theory.

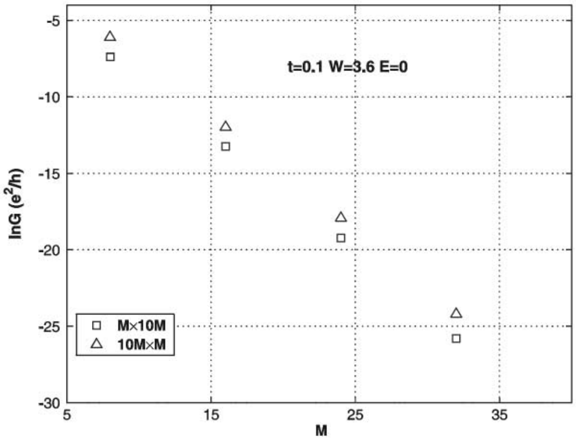

Fig. 1. The conductance G in units of e^2/h of an anisotropic system $M \times N$, versus M for $t = 0.1$ and $E = 0$. The aspect ratio 10 : 1 is chosen such that the sample dimension is approximately proportional to the localization length in that direction. Notice that G along the two directions stay close to each other while their over all magnitude changed many orders.

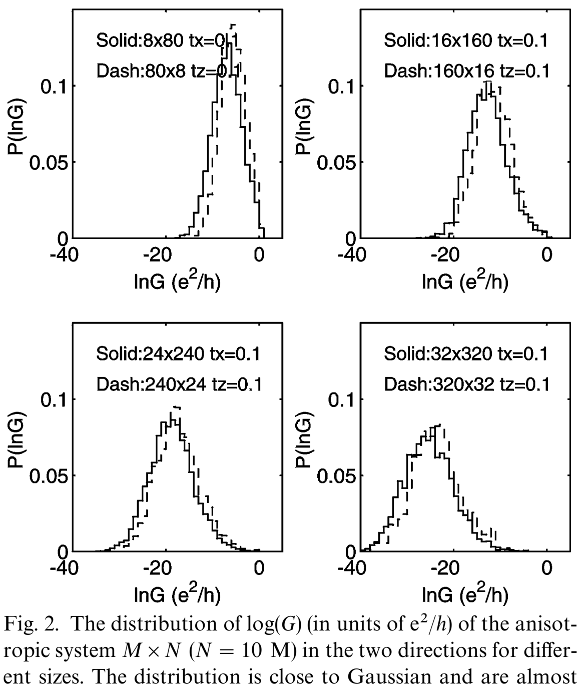

Fig. 2. The distribution of log(G) (in units of e^2/h) of the anisotropic system $M \times N$ ($N = 10$ M) in the two directions for different sizes. The distribution is close to Gaussian and are almost identical for the two directions.

We next examine the problem of the critical conductance at the localization–delocalization transition in an anisotropic system under a strong magnetic field, i.e., the integer quantum Hall plateau transition [24,25]. The tight-binding model was known to describe the quantum Hall transition well, at least in the lowest Landau band. For the purpose of investigating the scaling and the critical conductance, we have chosen a fixed magnetic field such that the flux per square is one-eighth of the flux quantum ($f = \frac{1}{8}$). The anisotropy is taken to be $t = 0.5$, and disorder strength is lowered to $W = 0.5$ to have a well-defined Lowest Landau band. From finite size scaling analysis, the arithmetic average of the conductance at the critical point for isotropic systems [22] was found to be $\langle G_c \rangle = 0.506e^2/h$, in agreement with the predicted value from analytical theory.

For anisotropic systems of a square geometry of $M \times M$, the conductance in the two direction will be different. In Figs. 3 and 4, we show the arithmetic averaged conductances in the two directions in the lowest Landau band. Again, the conductance peaks get narrower and narrower as the size of the system increases. This agrees with the conventional picture that only the state at the critical point is truly extended. Also, notice the much smaller conductance in the weak coupling direction ($\perp$) compared with that in the strong coupling direction ($\|$). An important property is the scaling of the conductance G as a function of the system size. According to the finite-size scaling idea, the conductance is expected to be determined solely by the ratio of the localization length to the system dimension M close to critical point. However, there is known irrelevant finite size correction such that the scaling is modified as

$$G(E,M) = G(E_c, M) f(\xi_\perp(E)/M, \xi_{\|}(E)/M), \quad (4)$$

where $\xi_\perp(E)$ and $\xi_{\|}(E)$ are the macroscopic localization lengths at energy E in the perpendicular (weak coupling) and the parallel (strong coupling) directions, respectively. $f(x, y)$ is a universal function. The size dependence of the conductance maximum $G(E_c, M)$ represents the irrelevant finite size corrections,

$$G_s(E_c, M) = G_c - aM^{-y_{\mathrm{irr1}}}. \quad (5)$$

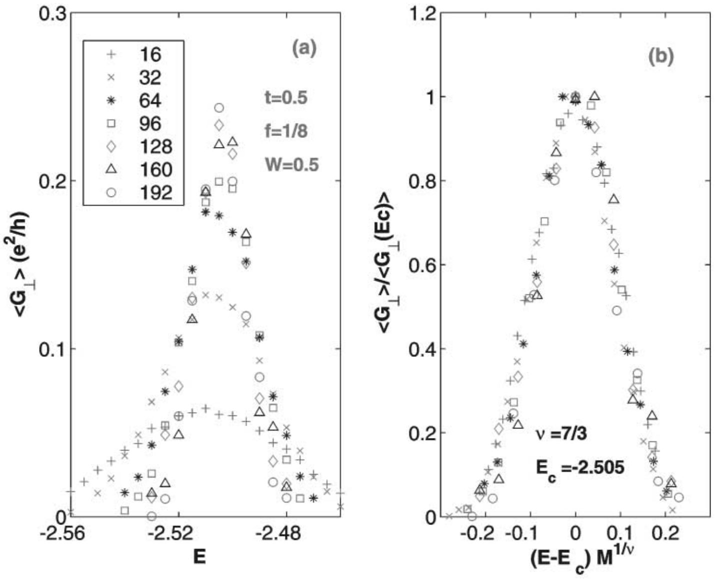

Fig. 3. Average conductance $\langle G_\perp \rangle$ in the lowest Landau band in the weak coupling direction. (a) Conductance versus energy for $M = 16, 32, 64, 96, 128, 160$, and 192. (b) Normalized conductance as a function of scaled variable $x = |E - E_c|M^{1/\nu}$ with $E_c = -2.505$ and $\nu = \frac{7}{3}$. The number of samples for each data point ranges from 100 for $M = 192$ and 10 000 for $M = 16$.

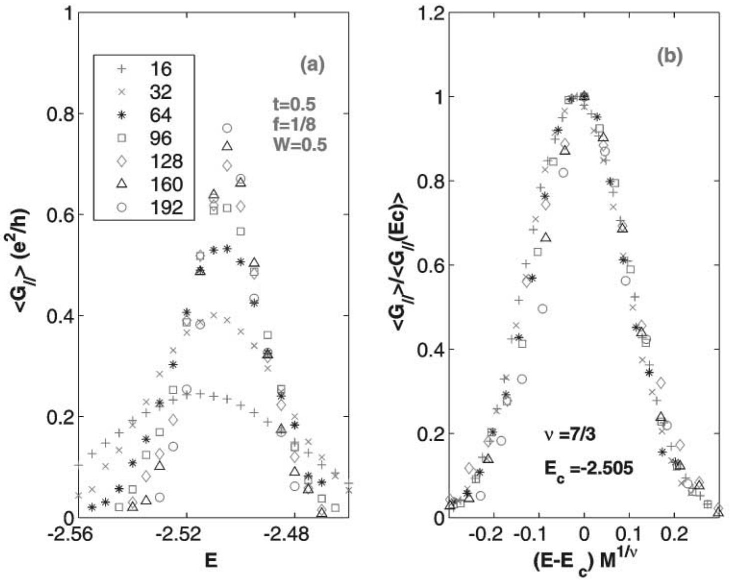

Fig. 4. Average conductance $\langle G_{||} \rangle$ in the lowest Landau band in the strong coupling direction. (a) Conductance versus energy for $M = 16, 32, 64, 96, 128, 160$, and 192. (b) Normalized conductance as a function of scaled variable $x = |E - E_c|M^{1/\nu}$ with $E_c = -2.505$ and $\nu = \frac{7}{3}$. The number of samples for each data point ranges from 100 for $M = 192$ and 10 000 for $M = 16$.

Utilizing $\xi_i \sim |E - E_c|^{-\nu}$, we obtain the expression

$$\begin{aligned} G_i(E, M, L) &= G_i(E_c, M) \\ &\times f(|E - E_c|^{-\nu}/M, |E - E_c|^{\nu}/L) \\ &= G_i(E_c, M)F(|E - E_c|M^{1/\nu}, M/L), \quad (6) \end{aligned}$$

where i denotes the two directions $\|$ and $\perp$. We have also used the fact that the aspect ratio M/L is a constant. Such a scaling analysis has been shown to work beautifully on isotropic systems [22] and the critical exponent ν was determined to be very close to the analytical value $\nu = \frac{7}{3}$. Should scaling exist in anisotropic system as well, then all of our data for different E and M would collapse on one curve, providing that the correct values of E_c and ν are chosen. The results of such a scaling procedure are shown in Figs. 3b and 4b for the arithmetic mean and Figs. 5b and 6b for the geometrical mean, respectively, with $E_c = -2.505$ and $\nu = \frac{7}{3}$. Scaling behavior is clearly established. Deviations seen on isotropic systems [32] for small M and higher energies due to the finite-size effect and the effect of mixing with higher bands, are not visible here.

Then critical conductance in the two directions, averaged over a large number of samples, are shown in Figs. 7 and 8 for the perpendicular and parallel direction, respectively. To extract the exact values for the critical conductances in the thermal dynamic limit, we have extended the procedure for the isotropic systems [22], with different irrelevant exponents in the two directions. However, we have not been successful in obtaining a good fit to the above form. There seem to be systematic deviations at small M. However, if we discard the data for $M = 8$ and 16, some reasonable fit is possible. We obtain $\langle G_{||c} \rangle = 1.04$ and $\langle G_{\perp c} \rangle = 0.33$ as the best fit for the two directions, respectively. This makes the geometrical mean $(\langle G_{||c} \rangle \langle G_{\perp c} \rangle)^{1/2} = 0.58$. Alternatively, we have also calculated the geometrical mean of the conductance from the two directions at each size and then fitted these values to the formula. This produces a best fit $\langle G_c \rangle_g = 0.60$, a very close value to the first method. The present result indicates that the geometrical mean of the critical conductances in the two directions are not universal. This again supports the conclusion that there are differences between the tight-binding model and

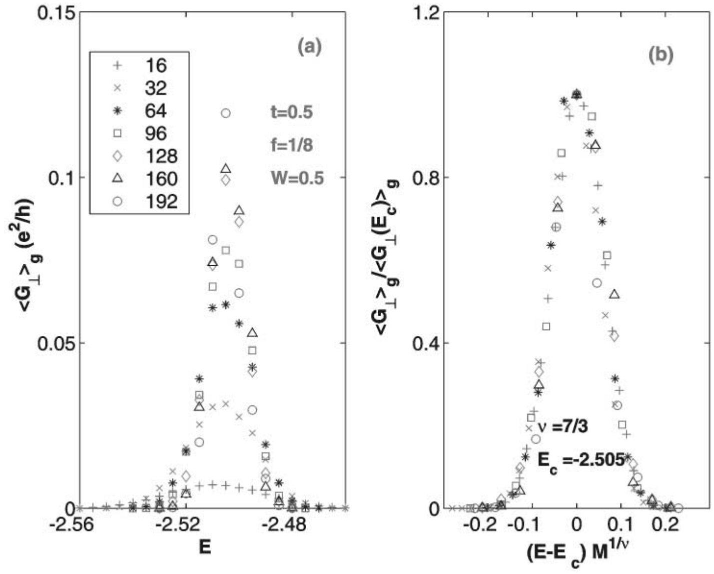

Fig. 5. Geometrically averaged conductance $\langle G_{\perp}\rangle$ in the lowest Landau band in the weak coupling direction. (a) Conductance versus energy for $M = 16$, 32, 64, 96, 128, 160, and 192. (b) Normalized conductance as a function of scaled variable $x = |E - E_c|M^{1/\nu}$ with $E_c = -2.505$ and $\nu = \frac{7}{3}$. The number of samples for each data point ranges from 100 for $M = 192$ and 10 000 for $M = 16$.

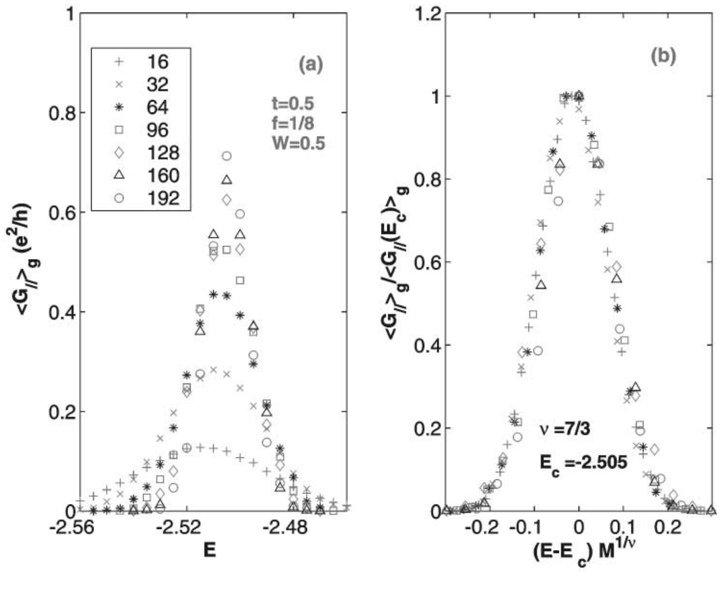

Fig. 6. Geometrically averaged conductance $\langle G_{||}\rangle$ in the lowest Landau Band in the strong coupling direction. (a) Conductance versus energy for $M = 16$, 32, 64, 96, 128, 160, and 192. (b) Normalized conductance as a function of scaled variable $x = |E - E_c|M^{1/\nu}$ with $E_c = -2.505$ and $\nu = \frac{7}{3}$. The number of samples for each data point ranges from 100 for $M = 192$ and 10 000 for $M = 16$.

the network model in terms of behaviors of the critical conductances. This needs to be examined closely.

The distribution of the conductance also shows interesting properties. For isotropic systems and at the critical point, the distribution is broad and

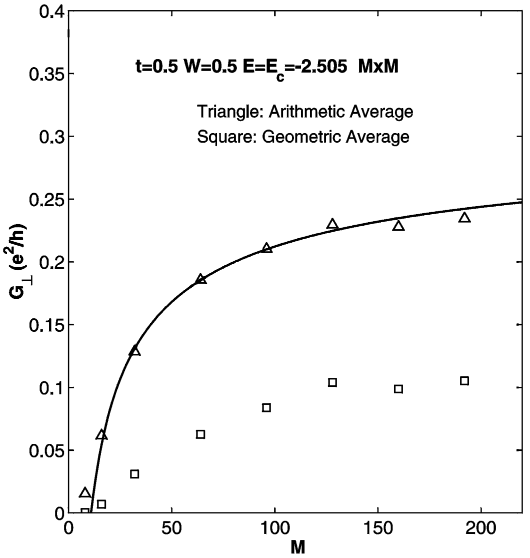

Fig. 7. Conductance at the critical point, $\langle G_{\perp}(E_c, M)\rangle$ in the weak coupling direction, as a function of system size M for square samples of $M \times M$. The extrapolated value for infinite size (see text) is $\langle G_{\perp c}\rangle = 0.32e^2/h$ for the arithmetic average. The error bars are smaller than the size of the symbols.

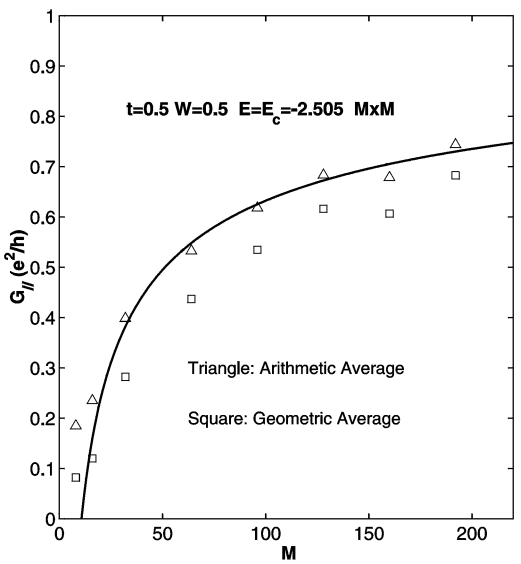

Fig. 8. Conductance at the critical point, $\langle G_{||}(E_c, M)\rangle$ in the weak coupling direction, as a function of system size M for square samples of $M \times M$. The extrapolated value for infinite size (see text) is $\langle G_{\perp c}\rangle = 1.04e^2/h$ for the arithmetic average. The error bars are smaller than the size of the symbols.

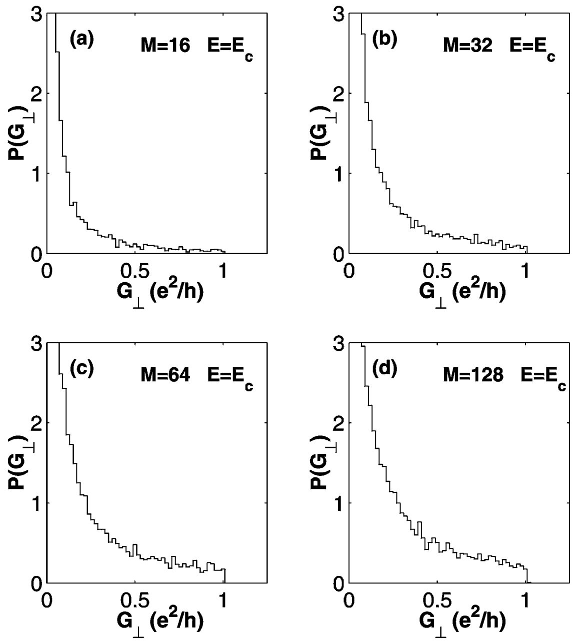

Fig. 9. Distribution of the conductance $G_{\perp}$ at the critical point $E_c = -2.505$ for different sample sizes. (a) $M = 16$, (b) $M = 32$, (c) $M = 64$, and (d) $M = 128$. Each size has more than 10 000 samples. Distributions at $M = 192$ (not shown here) is almost identical with that of $M = 128$ within the statistical fluctuation.

ranges between 0 and 1. Fluctuations, as measured by the standard deviation, is of the same order of magnitude as the average conductance itself. All these properties are still true in anisotropic systems as shown in Figs. 9 and 10 for the perpendicular and parallel directions, respectively. The distributions in the parallel direction (Fig. 10) resemble closely that of the isotropic system, having dip developing at small G with the increase of system sizes. The dip here is more pronounced though, and the maximum probability moves toward larger values of G. However, for the perpendicular direction the large weight at small G, a signature of the log-normal distribution appropriate for the localized states, persists even for the largest systems, $M = 192$. The overall weight nevertheless shifts toward larger values with increasing system sizes. At critical points, it has been proposed that the conductance distribution should be universal independent of the size of the system. This assertion is based on the fact that there is no length scale since

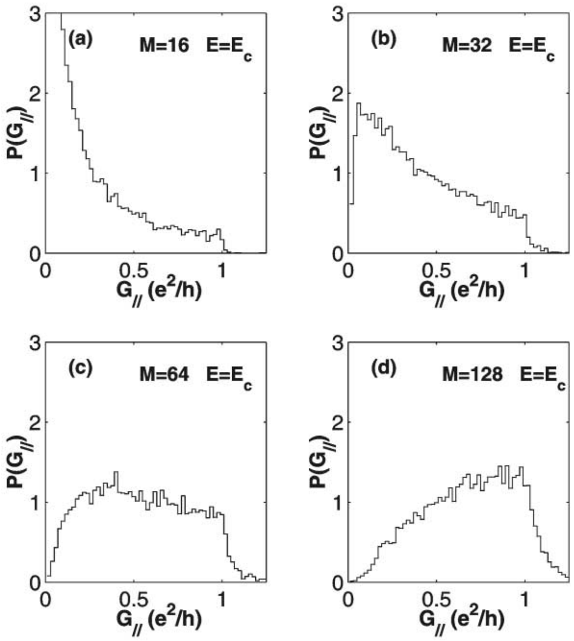

Fig. 10. Distribution of the conductance $G_{||}$ at the critical point $E_c = -2.505$ for different sample sizes. (a) $M = 16$, (b) $M = 32$, (c) $M = 64$, and (d) $M = 128$. Each size has more than 10 000 samples. Distributions at $M = 192$ (not shown here) shows more trend towards larger values of conductance.

the localization length diverges at the critical point. This is very interesting and needs to be investigated in the future.

4. Conclusions

In summary, we have investigated the scaling properties in two-dimensional systems with and without time-reversal symmetry breaking. For the system without any magnetic fields, we find that not only the averaged conductance $\langle G \rangle_g$ but also its distributions are approximately the same in the strong and weak coupling directions, if the system dimension is chosen to be proportional to the localization length in that direction. This is a strong confirmation of the scaling idea. In the localization–delocalization transition, we established the scaling of conductance in anisotropic systems around the critical point and confirmed that the critical exponent for the localization length is the same in both directions. The critical conductance in the two directions, extrapolated to infinite systems, are different, as expected. However, their geometrical products do not correspond to the value for isotropic systems, unlike in the network model. Again, this points out real differences between the tight-binding model and the network model so far as the critical conductance is concerned.

Acknowledgements

Ames Laboratory is operated for the US Department of Energy by Iowa State University under Contract No. W-7405-Eng-82. This work was supported by the Scalable Computing Laboratory of Ames Lab, the director for Energy Research, Office of Basic Energy Sciences.

References

[1] Q. Li, C.M. Soukoulis, E.N. Economou, G.S. Grest, Phys. Rev. B 40 (1989) 2825.
[2] I. Zambetaki, Q. Li, E.N. Economou, C.M. Soukoulis, Phys. Rev. Lett. 76 (1996) 3614.
[3] W. Xue, P. Sheng, Q.J. Chu, Z.Q. Zhang, Phys. Rev. Lett. 63 (1989) 2837.
[4] W. Xue, P. Sheng, Q.J. Chu, Z.Q. Zhang, Phys. Rev. B 42 (1990) 4613.
[5] Q.J. Chu, Z.Q. Zhang, Phys. Rev. B 48 (1993) 10 761.
[6] A.A. Abrikosov, Phys. Rev. B 50 (1994) 1415.
[7] A.G. Rojo, K. Levin, Phys. Rev. B 48 (1993) 16 816, and references therein.
[8] Y. Zha, S.L. Cooper, D. Pines, J. Phys. Chem. Solids 56 (1995) 1781.
[9] Y. Zha, Philos. Mag. B 74 (1996) 497.
[10] G. Kotliar et al., Europhys. Lett. 15 (1991) 655.
[11] V. Privman, M.E. Fisher, Phys. Rev. B 30 (1984) 322.
[12] C.-K. Hu, C.-Y. Lin, J.-A. Chen, Phys. Rev. Lett. 75 (1995) 193.
[13] C.-K. Hu, C.-Y. Lin, J.-A. Chen, Physica A 221 (1995) 80.
[14] Q. Li, S.J. Katsoprinakis, E.N. Economou, C.M. Soukoulis, Phys. Rev. B 58 (1997) R11227.
[15] B. Kramer, A. Mackinnon, Rep. Prog. Phys. 56 (1993) 1469.
[16] P. Wölfle, R.N. Bhatt, Phys. Rev. B 30 (1984) 3542.
[17] R.N. Bhatt, P. Wöffle, T.V. Ramakrishnan, Phys. Rev. B 32 (1985) 569.
[18] D. Vollhardt, P. Wölfle, in: W. Hanke, Yu.V. Kopaev (Eds.), Electronic Phase Transitions, Elsevier, New York, 1992, p. 38 (Chapter 1).

[19] W. Apel, T.M. Rice, J. Phys. C 16 (1983) L1151.
[20] A. Cohen, B. Shapiro, Int. J. Modern Phys. 6 (1992) 1243.
[21] K. Slevin, T. Ohtsuki, Phys. Rev. Lett. 78 (1997) 4083.
[22] X.-G. Wen, A. Zee, Int. J. Modern Phys. B 4 (1990) 437.
[23] M.P.A. Fisher, G. Grinsterin, S.M. Girvin, Phys. Rev. Lett. 64 (1990) 587.
[24] B. Hukestein, Rev. Modern Phys. 67 (1995) 357.
[25] R.E. Prange, S.M. Girvin (Eds.), The Quantum Hall Effect, Springer, New York, 1990.
[26] Z. Wang, B. Jovanovic, D.-H. Lee, Phys. Rev. Lett. 77 (1996) 4426.
[27] J.T. Chalker, P.D. Coddington, J. Phys. C 21 (1988) 2665.
[28] S.A. Kivelson, D.-H. Lee, S.-C. Zhang, Phys. Rev. B 46 (1992) 2223.
[29] J.L. Pichard, G. Andre, Europhys. Lett. 2 (1986) 477.
[30] D.S. Fisher, P.A. Lee, Phys. Rev. B 23 (1981) 685.
[31] E.N. Economou, C.M. Soukoulis, Phys. Rev. Lett. 46 (1981) 618.
[32] X. Wang, Q. Li, C.M. Soukoulis, Phys. Rev. B 58 (1998) 3576.