Transmission and reflection studies of periodic and random systems with gain

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The transmission (T) and reflection (R) coefficients are studied in periodic systems and random systems with gain. For both the periodic electronic tight-binding model and the periodic classical many-layered model, we obtain numerically and theoretically the dependence of T and R. The critical length of periodic system L_c^0, above which T decreases with the size of the system L while R approaches a constant value, is obtained to be inversely proportional to the imaginary part ε'' of the dielectric function ε. For the random system, T and R also show a nonmonotonic behavior versus L. For short systems ($L<L_c$) with gain $\langle \ln T\rangle=(l_g^{-1}-\xi_0^{-1})L$. For large systems ($L\gg L_c$) with gain $\langle \ln T\rangle=-(l_g^{-1}+\xi_0^{-1})L$. L_c, l_g, and ξ_0 are the critical, gain, and localization lengths, respectively. The dependence of the critical length L_c on ε'' and disorder strength W are also given. Finally, the probability distribution of the reflection R for random systems with gain is also examined. Some very interesting behaviors are observed. [S0163-1829(99)02809-X]

I. INTRODUCTION

While the study of localization of classical and quantum waves in random disordered media has been well understood,[1–4] recently, the wave propagation in amplifying random media has been pursued intensively.[5–13] Some interesting results have been predicted, such as, the localization length of a random medium with gain,[8] the sharpness of back scattering coherent peak,[5,10,14] the dual symmetry of absorption and amplification,[9] the critical size of the system,[6,8] and the probability distribution of reflection.[7] Numerically, two kinds of models are studied: one is the electronic tight binding model,[12,13] the other is the many-layered model of classical waves.[8] Theoretically, a lot of methods are used to get these results, such as the diffusion theory[6,11] and the transmission matrix method.[7] Most of these studies are for homogeneously random systems which are generated by introducing the disorder into the continuous system, and the medium parameter, such as the dielectric constant, is assumed to vary in a continuous way.[5,8] But the periodically correlated random systems which are generated by introducing the disorder into a periodic system, such as a photonic-band-structure, have not been studied adequately.

With gain, will such random systems with periodic background behave similar as the homogeneously random system? Both experimentally and theoretically, the study of such system is very important in understanding the propagation of light in random media. These type of photonic-band-structure systems are widely used in experiments.[3,15] Theoretically, just as John[2] argued, the localization of a photon is from a subtle interplay between order and disorder. For the periodically correlated random systems with gain, the periodic background plays the order role, and now its interplay with not only disorder but also with gain should be a very interesting new topic.

In this paper we address both the electronic tight binding model and the many-layered model of classical waves. We first compare the numerical results of periodic amplifying system with what we can predict theoretically by the transfer matrix method. It is surprising to get most of the universal properties, such as critical length and exponential decay of transmission, of homogeneously random system[8] from a periodic system too.

With the help of some theoretical arguments and numerical results, we suggest that the length $\xi_1=|1/\mathrm{Im}(K)|$, where K is the Bloch vector in a periodic system with gain, to replace the gain length $l_g=|1/\mathrm{Im}(k)|$ introduced in Ref. 8. This is more reasonable since the correlated scatterers in a periodic system can make the paths of wave propagation much longer in the system. We also think that it is actually the Bloch wave instead of the plane wave which propagates in the system. Then we introduce disorder into these periodic systems and calculate their properties. Our numerical simulations for both models show that periodically correlated random systems give similar behaviors as that of the homogeneously random systems studied previously. But in some cases, we get interesting results for the localization length ξ, the critical length L_c and the probability distribution $P(R)$ of reflection. All these results are related to the periodic background of such systems. We also examine the results of the transmission coefficient T for *short* ($L<L_c$) systems. Our numerical results show that the formula of the transmission coefficient of media with *absorption* can be generalized to the transmission coefficient of short systems with *gain,* if we replace the gain length l_g (or ξ_1) with the negative of the absorption length $-l_a$ in the formula. To explain our results of the critical length L_c, we compare the two basic theories for obtaining the critical length, the Letokhov theory[6] and the Lamb theory,[16] and we get some theoretical results of critical length which are in good agreement with our numerical results. The behavior of the distribution of the probability of reflection $P(R)$ is much more complex than the theoretical prediction of homogeneously random system.[7] We find that the periodic background influences strongly the general behavior of $P(R)$.

The paper is organized as follows. In Sec. II we introduce the two theoretical models we are studying. The results for the periodic systems with gain are presented in Sec. III, while in Sec. IV the results for the random systems with gain are given. Also in Sec. IV we present our theoretical and numerical results for the critical length L_c. In Sec. V the results for the probability distribution of reflection coefficient R for both models are presented. Finally, Sec. VI is devoted to a discussion of our results and give some conclusions.

II. THEORETICAL MODELS

A. Many-layered model of classical wave

Our periodic many-layered model of classical wave consists of two types of layers with dielectric constant $\varepsilon_1=\varepsilon_0-i\varepsilon''$ and $\varepsilon_2=\chi\varepsilon_0-i\varepsilon''$ and thicknesses $a=95$ nm and $b_0=120$ nm, respectively, where the negative part of dielectric constant, i.e., $\varepsilon''>0$, denotes the homogeneous amplification of the field. We have tried a lot values for χ, such as 1.5, 2, 3, 5, 6, and get no essential difference in our results for different values. In this paper we choose $\chi=2$, i.e., $\mathrm{Re}(\varepsilon_1)=\varepsilon_0=1$ and $\mathrm{Re}(\varepsilon_2)=2\varepsilon_0=2$. The system has L cells. Each cell is composed by two layers with dielectric constant ε_1 and ε_2 respectively. Without gain, we obtain that the wavelength range of the second band of this periodic system is from 247 nm to 482.6 nm (the first band has a range from 592 nm to infinite). So we choose the wavelength 360 nm to represent band center, the wavelength 420 nm as a general case, and the wavelength 470 nm to represent the band edge.

To introduce disorder, we choose the width of second layer of the nth cell to be random variable $b_n=b_0(1+W\gamma)$, where W describes the strength of randomness and γ is a random number between $(-0.5,0.5)$. The whole system is embedded in a homogeneous infinite material with dielectric constant equal to ε_0.

For the 1D case, the time-independent Maxwell equation can be written as

$$\frac{\partial^2 E(z)}{\partial z^2}+\frac{\omega^2}{c^2}\varepsilon(z)E(z)=0. \tag{1}$$

Suppose that in the medium with dielectric constant ε_1 and the medium with dielectric constant ε_2, the electric field[8] is given by the following expressions:

$$E_{1n}(z)=A_n e^{ik(z-z_n)}+B_n e^{-ik(z-z_n)},$$

$$E_{2n}(z)=C_n e^{ik(z-z_n)}+D_n e^{-ik(z-z_n)}. \tag{2}$$

Using the appropriate boundary condition (continuity of the electric field E and of the derivative of E at the interface), we obtain that

$$\begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}=(M_n)\begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{3}$$

where

$$(M_n)=\begin{pmatrix} e^{-ika}\left[\cos(qb_n)-\frac{i}{2}\left(\frac{k}{q}+\frac{q}{k}\right)\sin(qb_n)\right] & -\frac{i}{2}e^{-ika}\left(\frac{k}{q}-\frac{q}{k}\right)\sin(qb_n) \\ \frac{i}{2}e^{ika}\left(\frac{k}{q}-\frac{q}{k}\right)\sin(qb_n) & e^{ika}\left[\cos(qb_n)+\frac{i}{2}\left(\frac{k}{q}+\frac{q}{k}\right)\sin(qb_n)\right] \end{pmatrix}, \tag{4}$$

where $k=(\omega/c)\sqrt{\varepsilon_1}$ and $q=(\omega/c)\sqrt{\varepsilon_2}$.

From the product of these matrices, $M(L)=\Pi_1^L M_n$, we can obtain the transmission and reflection amplitudes of the sample, $t(L)=1/M_{11}$ and $r(L)=M_{21}/M_{11}$. For each set of parameters (L,W,ε''), the reflection coefficient $R=|r|^2$ and the transmission coefficient $T=|t|^2$ are obtained from a large number of random configurations. We have used 10 000 configurations to calculate the different average values of R and T, and 1 000 000 configurations to obtain $P(R)$. Our numerical results show that the localization length for a system without gain behaves $\xi_0\propto 1/W^2$ for this model, and are in agreement with previous workers.

B. Electronic tight-binding model

For the electronic tight-binding model, the wave equation can be written as

$$\begin{pmatrix} \phi_{n+1} \\ \phi_n \end{pmatrix}=(M_n)\begin{pmatrix} \phi_n \\ \phi_{n-1} \end{pmatrix}, \tag{5}$$

where

$$(M_n)=\begin{pmatrix} E-\epsilon_n & -1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

$\epsilon_n=W\gamma-i\eta$, where W describes the strength of randomness, γ is a random number between $(-0.5,0.5)$, $\eta>0$ corresponds to amplification and ϕ_n is the wave function at site n. The length L of the system is the total lattice number of the system. The system is embedded in two identical semi-infinite perfect leads on either side. For the left and the right sides, we have $\phi_0=1+r(L)$ and $\phi_{L+1}=t(L)e^{ik(L+1)}$. We can obtain reflection amplitude $r(L)$ and transmission amplitude $t(L)$ by the products of matrices, $M(L)=\Pi_1^L M_n$.

$$t(L)=\frac{-2i\sin(k)}{M_{11}e^{-ik}+M_{12}-M_{22}e^{ik}-M_{21}}e^{-ik(L+1)},$$

$$r(L)=\frac{M_{21}e^{ik}+M_{22}-M_{11}-M_{12}e^{-ik}}{M_{11}e^{-ik}+M_{12}-M_{21}-M_{22}e^{ik}}e^{-ik}, \tag{7}$$

where $k=\arccos(E/2)$.

When $W=0$ and without gain, the model is a periodic one with only one band spanning in energy between -2 and 2.

Notice that the hopping matrix elements in Eq. (6) are equal to one, which is our unit of energy. So we choose $E=0$ to represent band center, $E=1$ as a general case, $E=1.8$ to represent band edge.

Similar as the many-layered model, for each set of parameters (L,W,η), 10 000 random configurations were used to obtain a average value of R and T, and one million random configurations for $P(R)$. Theoretical and numerical results give that the localization length for a system without gain behaves $\xi_0 \propto 1/W^2$, in agreement with previous workers.

III. PERIODIC SYSTEMS

Almost all the properties of the periodic systems of both the many-layered model and the tight-binding model can be predicted theoretically.

A. Classical many-layered model

For long systems $(L \gg L_c^0)$ of the many-layered model we have that

$$\lim_{L\to\infty} \frac{\partial \ln T}{\partial L} = -\xi_1^{-1} = 2\,\mathrm{Im}(K) \propto \varepsilon'', \tag{8}$$

where K is the Bloch vector which is a complex number now, and satisfies $\cos K = \cos(ka)\cos(qb_0) - \frac{1}{2}(k/q + q/k)\sin(ka)\sin(qb_0)$. Because $\mathrm{Im}(K)<0$, the transmission coefficient T is decaying exponentially for a long system.

For a short system, $L<L_c^0$, we have

$$\frac{\partial \ln T}{\partial L} = 1/\xi_1' \simeq 2|C||\mathrm{Im}(K)| \simeq 2|\mathrm{Im}(K)|, \tag{9}$$

where $C=-[\sin(ka)\cos(qb_0)+\cos(ka)\sin(qb_0)]/[2\sin(K)]$, and $|C|$ is larger than but very close to 1 when wavelength is at the band center, and become bigger when wavelength approached the band edge.

So the slope of $\ln T$ vs L for a short periodic system is almost same as the negative value of the slope for the long system. The slopes of $\ln T$ at both sides of the maximum are approximately symmetric. In Fig. 1(a), we can see that, when $L<L_c^0$, T increases vs L with the slope $1/\xi_1'$, and get to a maximum at L_c^0 and decays exponentially when $L>L_c^0$ with the slope $1/\xi_1$.

From the behavior of the theoretical expressions of T or R, when T or R goes to infinite, we can obtain analytically that L_c^0 is given as

$$L_c^0 \simeq \xi_1 \ln\left(\frac{|C|+1}{|C|-1}\right), \tag{10}$$

where C is same as defined above in Eq. (9), and $|C|$ is close to one. From the property of $|C|$ discussed above, we can see that $L_c^0 \gg \xi_1$ at the band center , and becomes smaller when wavelength approaches the band edge. The value of $|C|$ is almost independent of gain, so L_c^0 is parallel to $1/\varepsilon''$ or ξ_1. We have shown that $L_c^0\varepsilon''$ is almost a constant for a given wavelength, and our numerical results agree very well with the theoretical prediction.

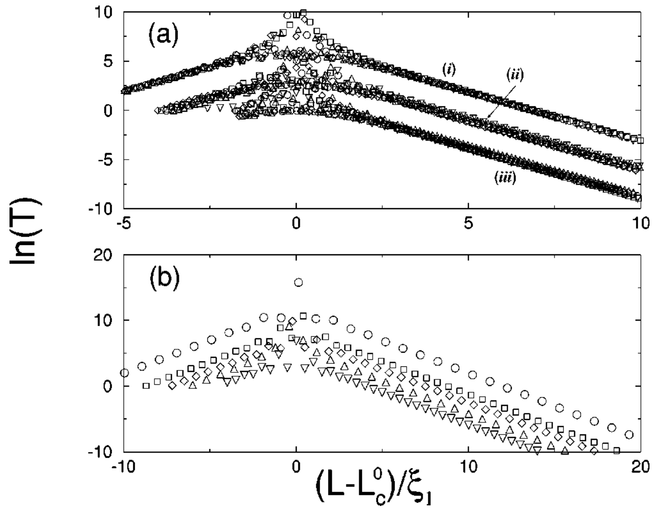

FIG. 1. The logarithm of the transmission coefficient T versus $(L-L_c^0)/\xi_1$, where L_c^0 is the critical length and ξ_1 is the gain length of periodic systems. (a) For the periodic many-layered model, (i), (ii), and (iii) are the values at three representable wavelengths: $\lambda = 360$ nm (band center), 420 nm (general), and 470 nm (band edge), respectively. The different symbols represent values obtained from different gains, $\varepsilon''=-0.001$, -0.002, -0.005, -0.001, and -0.1. (b) For the periodic tight-binding model with $E=0$ for different gains $\eta=0.01$, 0.05, 0.1, 0.2, and 0.5.

The reflection coefficient gets to a maximum value at L_c^0 too, and fluctuates a lot with the size L of the system. When L approaches infinity, R reaches a saturated value. The saturated value of R is given by

$$\lim_{L\to\infty} R = R_0 \simeq \frac{|(k/q-q/k)\sin(qb)|^2}{|\sin(K)(|C|-1)|^2}. \tag{11}$$

So R_0 is almost independent of gain or ξ_1. Figure 2 shows that indeed R increase when $L<L_c^0$, gets to its maximum

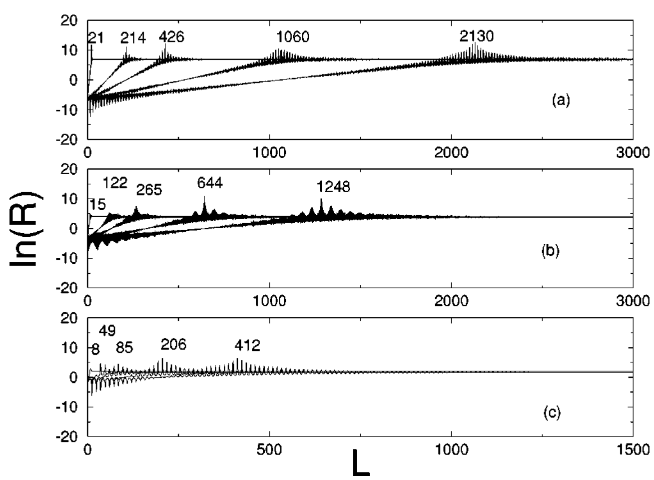

FIG. 2. The logarithm of the reflection coefficient R versus L for the periodic many-layered model. (a), (b), and (c) are values of three representative wavelengths $\lambda = 360$ nm, 420 nm, and 470 nm, respectively. From right to left, the numbers on the peaks are the values of L_c^0, corresponding to different gains $\varepsilon''=-0.001$, -0.002, -0.005, -0.01, and -0.1. Notice that the saturated value of R is independent of ε'' for the three wavelengths studied.

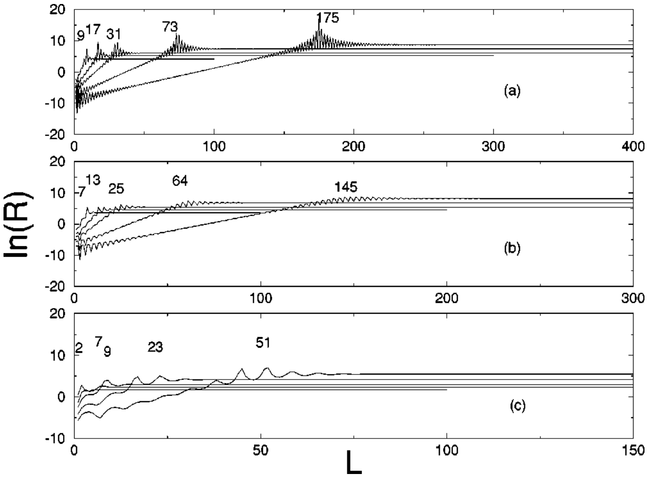

FIG. 3. The logarithm of the reflection coefficient R versus L for the periodic tight-binding model; (a), (b), and (c) are values of three representative energies: $E=0$, 1, and 1.8. From right to left, the numbers on the peaks are the values of L_c^0 corresponding to different gains $\eta=0.01$, 0.05, 0.1, 0.2, and 0.5. Notice that the saturated value of R for each E depends on η.

value and fluctuate violently at L_c^0, then approaches a saturated value which is almost independent of gain.

B. Electronic tight-binding model

For the electronic tight-binding model, when $E=0$, the $\ln T$ for the long system can be obtained by the use of Eq. (7):

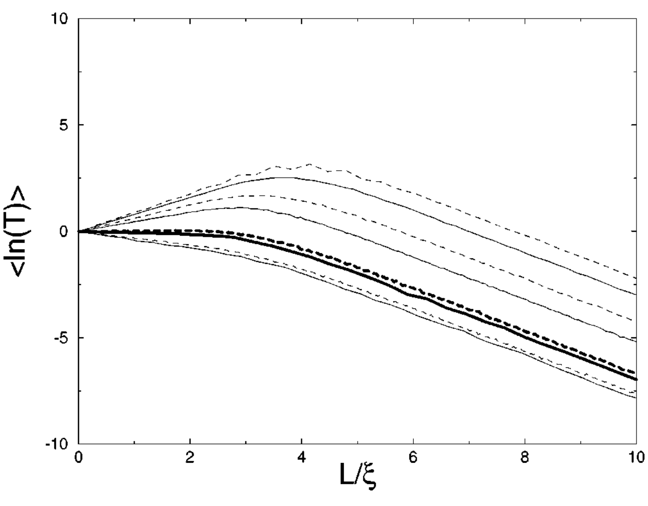

FIG. 4. The average values of the logarithm of T versus L/ξ. The results of the random many-layered model, with $\lambda=360$ nm and $W=0.2$, are shown by solid lines. Lines from lower to higher correspond to different gains $\varepsilon''=0.0005$, 0.001, 0.005, and 0.01. When ε'' is equal to 0.001, $1/\xi_1$ is almost the same as $1/\xi_0$, so $\langle \ln T\rangle$ is almost horizontal for small L, as shown by the wide solid line. For $\varepsilon''>0.001, 1/\xi_1>1/\xi_0$, and for $\varepsilon''<0.001, 1/\xi_1<1/\xi_0$. Results for the random tight-binding model, with $E=0$ and $W=1$, are shown by dashed lines. Lines from lower to higher correspond to $\eta=0.01$, 0.02, 0.08, and 0.3. When η is equal to 0.02, $1/\xi_1$ is almost the same as $1/\xi_0$, so $\langle \ln T\rangle$ is almost horizontal for small L, as shown by the wide dashed line. For $\eta>0.02, 1/\xi_1>1/\xi_0$, and for $\eta<0.02, 1/\xi_1<1/\xi_0$.

$$\lim_{L\to\infty}\frac{\partial \ln T}{\partial L}=-1/\xi_1\simeq-\eta. \tag{12}$$

Similarly as the classical many-layered model, for short system of tight-binding model we have

$$\frac{\partial \ln T}{\partial L}=1/\xi_1'\simeq\eta. \tag{13}$$

So the slope symmetry of $\ln T$ at both sides of L_c^0 still exists. In Fig. 1(b), we can see a similar behavior as in Fig. 1(a), when $L<L_c^0$, $\ln T$ change vs L with the slope of $1/\xi_1'$ and gets maximum at L_c^0, then it begin to decay exponentially with a slope of $1/\xi_1$.

Assuming that the theoretical expression of T, given by Eq. (7), is infinite, we can obtain that L_c^0 is given by

$$L_c^0\simeq\frac{2}{\eta}(\ln 4-\ln\eta)\simeq 2\xi_1\ln(4\xi_1). \tag{14}$$

We also shown that $L_c^0\eta+2\ln\eta$ vs η, for $E=0$, is a constant for different gain and indeed find out that the theoretical prediction given by Eq. (14) agree very well with the numerical results.

The reflection coefficient R approaches a saturated value as L goes to infinite, but the saturated value of R_0 is not a constant independent of the gain as in the case of classical many-layered model. This is clearly seen in Fig. 3 where we plot $\ln R$ vs L. Notice that the $\ln R$ curves increase vs L when $L<L_c^0$, get to a maximum at L_c^0, and then approach a saturated value when L goes to infinity. Similar results were obtained for $E\neq 0$.

IV. RANDOM SYSTEMS

In Fig. 4, we give the general behavior of average value $\langle \ln T\rangle$ vs L for both models. We can see the different behaviors for $L<L_c$ and $L>L_c$. When $1/\xi_1>1/\xi_0$ and $L<L_c$, $\langle \ln T\rangle$ increase vs L from origin with a slope which is defined as $1/\xi'$ and when $L>L_c$, $\langle \ln T\rangle$ decrease vs L with a slope $-1/\xi$. But when $1/\xi_1<1/\xi_0$, $\langle \ln T\rangle$ will decrease monotonically, at first with the slope $1/\xi'=-1/|\xi'|$, at L_c, there are a turning point and slope changes to $-1/\xi$. We will study the values of ξ', ξ and L_c in this section.

It was first suggested by Zhang[8] that the localization length ξ of a long random system with gain will become smaller than the localization length ξ_0 of the random system without gain. In particular he suggested that

$$\frac{1}{\xi}=\lim_{L\to\infty}\frac{\partial \ln T}{\partial L}=\frac{1}{\xi_0}+\frac{1}{\xi_1}, \tag{15}$$

where ξ_0 is the localization length of the system without gain, l_g is replaced by ξ_1 in the original formula of Zhang because of the periodic background of our systems.

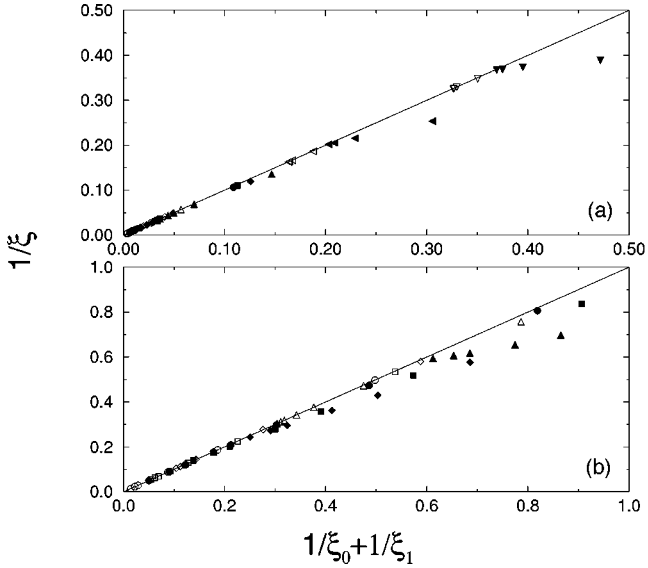

FIG. 5. $1/\xi$ versus $1/\xi_0+1/\xi_1$, where ξ is the localization length for a system with gain, ξ_0 is the localization length of a system with disorder but with zero gain, and ξ_1 is the gain length. (a) For the random many-layered model, empty symbols are of wavelength $\lambda=360$ nm and filled symbols are of wavelength $\lambda=470$ nm. Different symbols represent different sets of parameters of disorder and gain: $W=0.05$, 0.1, 0.2, and 0.5; $\varepsilon''=0.001$, 0.002, 0.005, 0.01, 0.05, and 0.1. (b) For the random tight-binding model, empty symbols are of energy $E=0$, and filled symbols are of energy $E=1.8$. Different symbols represent different sets of parameters of disorder and gain: $W=0.5$, 0.8, 1, 1.5, 2, 3, and 5; $\eta=0.01$, 0.05, 0.1, and 0.3.

We have numerically calculated $1/\xi$ for different cases of disorder, gain and frequency(energy), and compare it with $1/\xi_0+1/\xi_1$, as shown in Figs. 5(a) and 5(b) for the many-layers model and the tight-binding model respectively. For most of the cases, Eq. (15) is a very good formula. Only when wavelength is on band edge *and* when both gain and randomness are very strong, we can see that the numerical results deviate from the theoretical prediction, which is the solid line in both Figs. 5(a) and 5(b).

For a short system ($L<L_c$), the behavior of $\langle \ln T\rangle$ vs L is quite different from that of a long system as shown in Fig. 4. Freilikher *et al.* and Rammal and Doucot[17] obtained that the transmission coefficient of a random system with *absorption* is given by

$$\langle \ln T\rangle=\left(-\frac{1}{l_a}-\frac{1}{\xi_0}\right)L, \tag{16}$$

where l_a is the absorption length and ξ_0 is the localization length. For a medium with *gain*, can we just substitute the $-l_a^{-1}$ with l_g^{-1} in Eq. (16) to get the following equation?

$$\frac{\langle \ln T\rangle}{L}=\frac{1}{\xi'}=\left(\frac{1}{l_g}-\frac{1}{\xi_0}\right). \tag{17}$$

Because of the periodic background of our models, we use ξ_1 to replace l_g in our calculations.

So far there is no independent verification for this conclusion. After substituting ξ_1 for l_g, our numerical results show that Eq. (17) is correct for *short* systems with *gain* for both models. When the strength of the disorder is a constant , so ξ_0 is a constant, according to Eq. (17), $1/\xi_1-1/\xi'$ should be equal to $1/\xi_0$ and be constant as the gain varies. We have checked this prediction and find indeed that the numerical values are almost the same as the ones predicted theoretically.

From Eq. (17), we can predict the basic features of the length dependence of $\langle \ln T\rangle$ shown in Fig. 4. When $1/\xi_1>1/\xi_0$, $\langle \ln T\rangle$ will increase with L, and will reach a maximum value when L gets to L_c. But if $1/\xi_1<1/\xi_0$, the $\langle \ln T\rangle$ will decrease monotonically, at first with a slope of $-|1/\xi_1-1/\xi_0|$ from the origin, at L_c the curve has a turning point and the slope changes to $-|1/\xi_1+1/\xi_0|$. If $1/\xi_1\simeq 1/\xi_0$, the curve is almost horizontal for small L and begins to decrease with a slope $-1/\xi$ at the critical length. This behavior is exactly shown in Fig. 4.

The critical length L_c is one of the most important parameters of a random system with gain. For a random system, one of the most important theories is the Letokhov theory.[6] Zhang[8] generalized the theory and used the no-gain localization length ξ_0 to replace the diffusion coefficient D in the Letokhov theory and obtain that the critical length $L_c\simeq\sqrt{\xi_0 l_g}$, so that we can clearly see localization effects in the system. But as shown above, there is a finite critical length in periodic system when the no-gain localization length ξ_0 goes to infinite, so there must be other mechanisms for determining the critical length in those systems. We find that when the localization effect is strong enough so that the no-gain localization length $\xi_0 \ll (L_c^0)^2/\xi_1$, then the results of the Letokhov theory are quite good. But when the system randomness is weak so that ξ_0 is larger than $(L_c^0)^2/\xi_1$, then the Letokhov theory results are not correct, and we have to use other theories, such as the Lamb theory,[16] which is well known is laser physics, to determine L_c. Next we will compare the Letokhov theory with the Lamb theory, and find the expressions of the critical length in different cases.

According to the Letokhov theory[6,11] the field in the system satisfies

$$\frac{\partial\phi(\vec{r},t)}{\partial t}=D\nabla^2\phi(\vec{r},t)+\frac{c\,\phi(\vec{r},t)}{l_g}, \tag{18}$$

where D is the diffusion coefficient and c is the speed of the wave.

Considering the relaxation after long time, the solution[11] of Eq. (18) is

$$\phi(\vec{r},t)\propto e^{-t[D(\pi/L)^2-c/l_g]},$$

$$\frac{\partial\phi(\vec{r},t)}{\partial t}=-D\left(\frac{\pi}{L}\right)^2\phi(\vec{r},t)+\frac{c\,\phi(\vec{r},t)}{l_g}. \tag{19}$$

When $L=L_c=\pi\sqrt{Dl_g/c}$, the system is at a critical point. If $L<L_c$ the field will decay vs time, but if $L>L_c$ then the field in the system will become stronger and stronger with time.

We can clearly see that the physical meaning of L_c is the balance point of the gain and loss in the system. When the L is less than L_c, the photon escaping rate, which is determined by $D\pi^2/L^2$, is larger than the photon generating rate, which is determined by c/l_g of the system, so the photons generated by the stimulated emission can escape from the system instantaneously and the system can get to the static state after a long time. If L is larger than L_c, gain is larger than loss, and photons will be accumulated in the system.[11] Based on the Letokhov theory and the weak localization theory[1,2] results, Zhang[8] generalized the L_c to be $L_c \simeq \sqrt{\xi_0 l_g}$ since $D=\frac{1}{3}lc$, where l is the mean free path and $\xi_0=(2\sim 4)l$.

In our models, considering the periodic background, we substitute ξ_1 for l_g first. But when the disorder becomes weaker and weaker, the system become almost periodic, ξ_0 goes to infinite, L_c goes to L_c^0 instead to infinite. How one can explain this behavior of L_c? The Lamb theory can give a theoretical explanation of it. In the Lamb theory, a phenomenological parameter Q(L), the quality factor which generally is a function of system length L, is introduced to show the energy loss rate of the system (also can be thought as the photons loss rate of the system). In the Lamb theory, the magnitude of the electric field in a linear medium satisfies the following equation:

$$\frac{\partial |E(t)|}{\partial t}=-\frac{\omega}{2Q(L)}|E(t)|+\frac{c}{l_g}|E(t)|. \tag{20}$$

At the critical condition, the gain term and loss term are equal. We have

$$\frac{\omega}{2Q(L_c)}=\frac{c}{l_g}. \tag{21}$$

If we compare Eq. (20) with the solution of Letokhov theory, Eq. (19), we can find the similarity between them. This similarity is from the same physical principle, the interplay of loss and gain in the system. From Eq. (20) we can see that the gain term is same as the one given by the Letokhov theory, the only difference is from loss term. Generally, $\omega/2Q$ is a function of the system length, e.g., for Fabry-Pérot interferometer $\omega/2Q\propto 1/L$.[16,18] For periodic system $Q=Q_p$, we have $\omega/2Q_p\propto 1/L$ too. From the balance of gain and loss, we can get $L_c^0\propto\xi_1$ in agreement with our results presented in Sec. III. This means that in a periodic system the rate of loss is not infinite, although the no-gain localization length goes to infinity. The rate is determined by the Q_p of the system and we can get a finite L_c^0 correspondingly. From the L_c^0 obtained above and the critical condition given by Eq. (21), we have that the quality factor of the periodic system is given by

$$Q_p=\frac{\omega\xi_1}{2cL_c^0}L, \tag{22}$$

which is independent of no-gain localization length ξ_0.

For a random system, things are a little more difficult. The theory of Letokhov does not give the detailed information of localized modes but it gives a localization related quantity D, the diffusion coefficient. According to the localization theory, D is directly related with localization length ξ_0, just as Zhang discussed.[8] Based on the correct results of the Letokhov theory in the strong localization case, we can assume that when the disorder is strong enough $\xi_0 \ll L_c^{0^2}/\xi_1$, the localization effects will dominate the escape rate of photons of the system (Our numerical results shown in Fig. 6 support this assumption). Lamb theory gives that the Q of a strong random system is determined by the localization effect. By comparing the corresponding terms in Eq. (19) and Eq. (20), we obtain that

$$Q\simeq Q_l=\frac{\omega L^2}{\pi^2 D}=\frac{\alpha\omega L^2}{c\xi_0}, \tag{23}$$

where the subscript l is for localized modes, α is a constant of the order of unity and depends on the ratio of D and ξ_0 according to the localization theory. For both of the models studied here, we find the α can be chosen to be equal to 0.7. From this we can get that the critical length $L_c=\sqrt{1/\alpha\xi_0\xi_1}\simeq\sqrt{\xi_0\xi_1}$ which is consistent with the Letokhov theory. Equation (23) is a very interesting result for laser physics because it is obtained by the comparison of the Letokhov and Lamb theories, and it directly gives the relationship of the quality factor Q of a random system with the no-gain localization length ξ_0 of the system.

In the weak disorder limit $\xi_0 \gg L_c^{0^2}/\xi_1$, $Q\to Q_p$ and $L_c\to L_c^0$. In strong disorder limit $\xi_0\ll L_c^{0^2}/\xi_1$, $Q\to Q_l$ and $L_c\to\sqrt{\xi_0\xi_1}$. For cases where ξ_0 is comparable to $L_c^{0^2}/\xi_1$, both the effects of periodic background and randomness will be important to determine the quality factor of such a system. Considering the Q as the photon-resistance in the system, and if we assume that both effects are *independent* with each other, we have that the total quality factor of the system to be

$$Q=Q_p+Q_l=\frac{\omega}{c}\left(\frac{\xi_1 L}{L_c^0}+\frac{\alpha L^2}{\xi_0}\right). \tag{24}$$

From the critical condition, Eq. (21), we have that

$$L_c=-\frac{\xi_0\xi_1}{2\alpha L_c^0}+\sqrt{\left(\frac{\xi_0\xi_1}{2\alpha L_c^0}\right)^2+\frac{\xi_0\xi_1}{\alpha}}. \tag{25}$$

In Figs. 6(a) and 6(b), we compare the theoretical predictions given by Eq. (25) and by Zhang[8] with our numerically calculated results for the classical many-layered model and the electronic tight-binding model, respectively. Our numerical results shown in Figs. 6(a) and 6(b) strongly support Eq. (25) to be the correct expression of the critical length L_c for both the weak and the strong random limits. In some other cases, the deviation can be as large as fifteen percent which is still very good considering the many approximations that have been introduced in derivation of Eq. (25). One explanation for this deviation is that the two effects that were added in Eq. (24) are not totally *independent*, because the correlated scattering of the periodic background will affect both Q_p and Q_l.

V. PROBABILITY DISTRIBUTION OF REFLECTION COEFFICIENT

Pradhan and Kumar[7] first obtained the probability distribution of the reflection coefficient for a long system with randomness and gain, which is given by

$$P(x)=P\left(\frac{R-1}{2q}\right)=\left(\frac{2q}{R-1}\right)^2\exp\left(\frac{-2q}{R-1}\right), \tag{26}$$

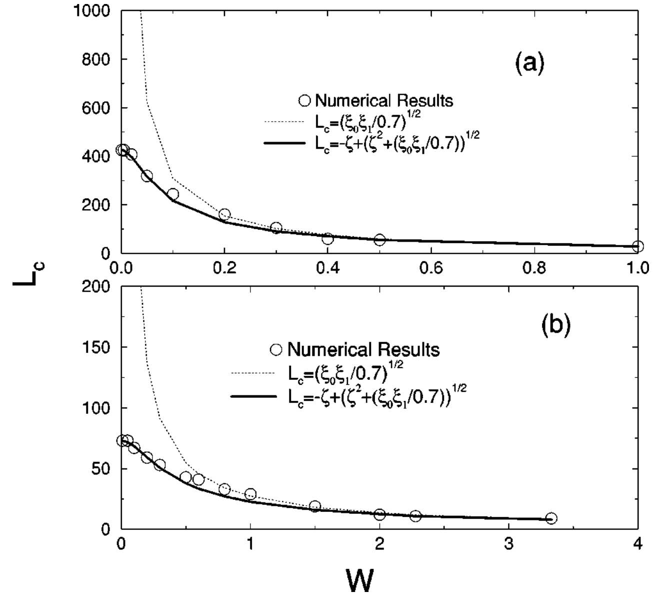

FIG. 6. The critical length L_c is plotted versus different random strengths W. (a) For the random many-layered model with $\lambda = 360$ nm, the dashed line and darkened line are the values obtained according to Zhang's formula and Eq. (25), respectively. (b) For the random tight-binding model with $E=0$ and $\eta=0.01$, the dashed line and darkened line are the values obtained according to Zhang's formula and Eq. (25), respectively. In both cases $\zeta = \xi_0\xi_1/2\alpha L_c^0$ and $\alpha=0.7$.

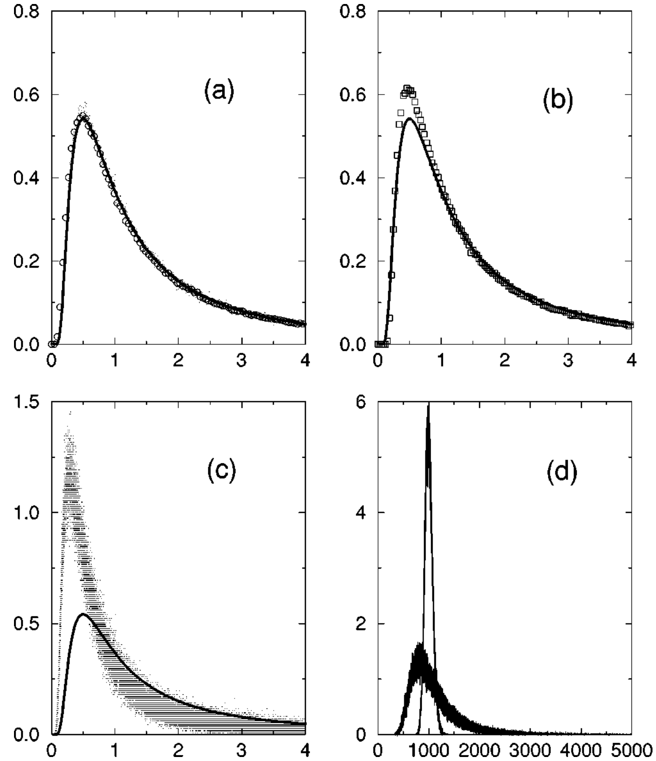

FIG. 7. Probability distribution of the reflection coefficient $P(x)$ versus x of the random many-layered system with gain at $\lambda = 300$ nm for $q=1.1$ and 17.7 (a); $q=0.163$ (b); $q=451$ (c), where $x=(R-1)/q$ and $q=\xi_0/\xi_1$. The solid curve given by the solid line in (a), (b), and (c) is the analytical result of Eq. (26). In (d), $P(R)$ versus R is plotted for two values of q: $q=1800$ (low one) and 7200 (high one). Notice that $P(R)$ approaches a delta-function distribution at R_0 when $q=7200$.

where $x=(R-1)/2q$ and $q=\xi_0/\xi_1$. We numerically calculated the $P(x)$ (or $P(R)$) for both models in cases that q changes drastically. Our numerical features of $P(x)(P(R))$ give some interesting results. According to Eq. (26), the maximum probability of $P(x)$ (or $P(R)$) should appear at 0.5 (or $R=q+1$), and the distribution has a long tail of large x (or R).

For the many-layered model, when the wavelength is near the band center ($\lambda=360$ nm), we have that $L_c^0/\xi_1 \simeq 7$ for different gains from Eq. (10). When q at the range of 0.01 to 50, and so $\xi_0 < L_c^{0^2}/\xi_1$, the $P(x)[P(R)]$ behaves as predicted by Eq. (26). When q is at the range of 50 to 500, the position of the maximum $P(x)(P(R))$ begins to shift left away from the point 0.5 (or $q+1$). When q increase further, such as to become close to a thousand, the maximum of $P(R)$ shift to the value of R_0, which is the saturated value of reflection for the periodic system, and $P(R)$ begin to change its shape into a delta function and the long tail disappears. This process is clearly shown in Fig. 7. It is reasonable to assume that when q is very large, the system is similar to a periodic system, so the $P(R)$ changes to a delta function which is the distribution of the reflection coefficient of the periodic system.

For the tight-binding model, when frequency is at band center ($E=0$), L_c^0/ξ_1 is not a constant as η changes. It has a range from 5 to 7. When q is at the range of 0.01 to 20 ($\xi_0 < L_c^{0^2}/\xi_1$), the $P(x)$ (or $P(R)$) behaves as predicted by Eq. (26). When q is at the range of 20–400, then the position of the maximum $P(x)(P(R))$ begins to shift left away from theoretical value 0.5 (or $q+1$). When q is larger than 400, $P(R)$ develops two peaks, one peak evolves from the original peak, the other one emerges at R_0. When q is even larger, such as thousands, then the original peak goes down and disappears, and the new peak become higher at R_0. At the same time the long tail disappears, the $P(R)$ also changes to a delta function at the position of R_0. All these changes are shown clearly in Fig. 8. In Ref. 13, they also got two peaks for $P(R)$, but they did not explain that the new peak is due to the periodic background of the system and that the delta function is at the position of R_0, the saturated value of periodic system. When q is very small, we obtain that the $P(x)$ (or $P(R)$) is almost the same as the one predicted by Eq. (26), quite different from the results of Ref. 13. We think that this difference is due to the fact that they have not renormalized their numerical results.

In summary, from our numerical results, we get the general behavior of $P(x)$ (or $P(R)$) for both models. When $\xi_0 < L_c^{0^2}/\xi_1$, the $P(x)$ (or $P(R)$) is same as the theoretically predicted one by Eq. (26). When ξ_0 is bigger than $L_c^{0^2}/\xi_1$, we must think about the effect of the periodic background and if ξ_0 is really very large, the periodic background will dominate the behavior of $P(R)$. We also find that at the band edge wavelength for many-layered model ($\lambda=470$ nm) or at the band edge energy for the tight-binding model (E=1.8) the effects of coherent scattering will be very strong and make the long paths of wave propagation more important. So we think it is this coherent scattering effect which makes the results of $P(x)$ (or $P(R)$) always different from the predic-

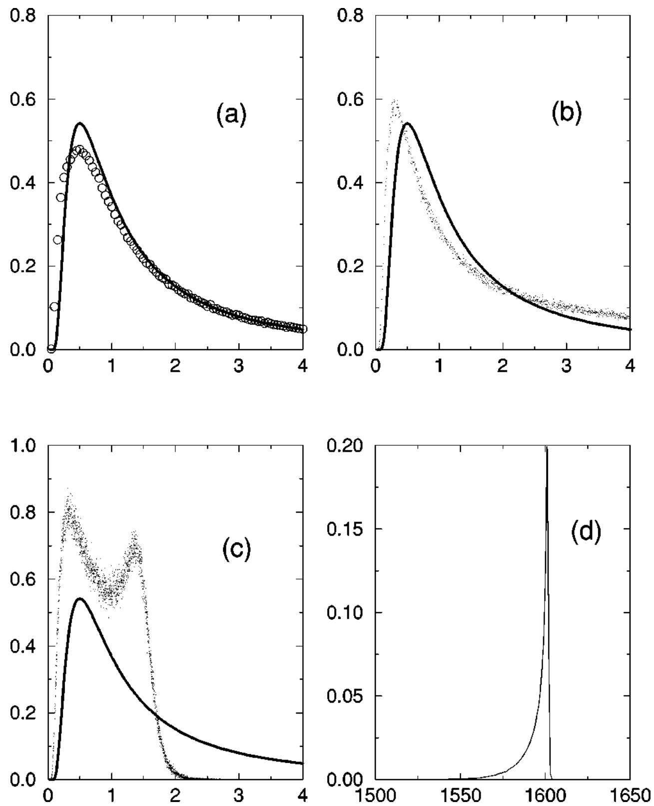

FIG. 8. Probability distribution of the reflection coefficient $P(x)$ versus x of the random tight-binding system with gain at $E=0$ for $q=1.0$ (a); $q=132$ (b); $q=525$ (c), where $x=(R-1)/q$ and $q=\xi_0/\xi_1$. The solid curve given by the solid line in (a), (b), and (c) is the analytical result of Eq. (26). In (d), $P(R)$ versus R is plotted for $q=5.25\times10^4$.$P(R)$ approaches a delta-function distribution at R_0 when $q=5.25\times10^4$.

tions given by Eq. (26) which is obtained for homogeneous random systems.

VI. CONCLUSION

In conclusion, we have studied the transmission and reflection coefficients in periodic or periodically correlated random systems with homogeneous gain. Theoretically, for periodic systems we predicted the behaviors of transmission and reflection coefficients, such as the slopes of long systems and of short systems and critical length by the transmission matrix method. For random systems, first the Zhang's formula of the localization length for long systems is checked. We find that only at the band edge and with very strong gain and strong disorder, there is obvious deviation from the theoretical prediction of the localization length with gain. For short systems, our numerical results show that our generalization of the formula of absorbing system is correct for amplifying systems. According to this generalization we can predict the behaviors of average values of logarithm of transmission coefficient $\langle \ln T\rangle$ from the value of $1/\xi'$, such as if it is positive then the $\langle \ln T\rangle$ will increase from origin at slope $1/\xi'$ and generate a peak at Lc and then start to decrease at slope $-1/\xi$; if it is negative then the $\langle \ln T\rangle$ will decrease monotonically and has a turning point at L_c with the slope change from $-1/|\xi'|$ to $-1/\xi$.

To explain the behavior of the critical length L_c which we got from our numerical results, we compare the Letokhov theory with the Lamb theory and give a general expression for the critical length considering both the effects of localization and periodic background. With this comparison, we also construct the relation of the quality factor Q of a random system with the localization length ξ.

We also study the probability distribution of the reflection coefficient $P(R)$ of random systems with gain. We find some new behaviors of $P(R)$ and give the criteria for the range of validity of the different behaviors and explain it by the influence of the periodic background too. The study of wave propagation in an amplifying random system is a challenging topic. There are still a lot of things to be done.[18]

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