Scaling properties of conductance at integer quantum Hall plateau transitions

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We investigate the scaling properties of zero-temperature conductances at integer quantum Hall plateau transitions in the lowest Landau band of a two-dimensional tight-binding model. Scaling is obeyed for all energy and system sizes with critical exponent $\nu \approx \frac{7}{3}$. The arithmetic average of the conductance at the localization-delocalization critical point is found to be $\langle G \rangle_c = 0.506 e^2/h$, in agreement with the universal longitudinal conductance $\langle \sigma_{xx} \rangle = \frac{1}{2} e^2/h$ predicted by an analytical theory. The probability distribution of the conductance at the critical point is broad with a dip at small G. [S0163-1829(98)01532-X]

The transitions between the integer quantum Hall (IQH) plateaus are believed to be a manifestation of the localization to delocalization transition in two-dimensional electron systems in the presence of a strong magnetic field. The phenomenon[1] is characterized at finite temperatures by the appearance of a conductance peak as the Hall conductance varies continuously between the precisely quantized values with the change of applied magnetic field.[1] The existence of both extended and localized states is required to describe this fascinating phenomenon. Following extensive experimental and theoretical investigation, a consistent picture has emerged on the nature of this remarkable transition. In a two dimensional disordered noninteracting electron system, extended states do not exist as a result of Anderson localization except at a singular energy near the center of each of the Landau subbands.[2,3] The localization length diverges at these critical energies as $\xi \sim |E - E_c|^{-\nu}$, signifying a continuous zero-temperature quantum phase transition. Electron conduction close to the transition is controlled by the ratio of the coherent length over the localization length.[4] As the system size increases or, equivalently, as the temperature decreases to zero for macroscopic samples, the system scales to stable Hall insulator fixed points characterized by $(\sigma_{xx}, \sigma_{xy}) = (0,n) e^2/h$.

Although a consensus has been reached on the general picture, several fundamental issues remain unsolved. One such issue, the universality of the transition, has attracted much attention recently.[5] Based on Chern-Simons formalism, Kivelson, Lee, and Zhang[6] predicted a transition between quantized plateaus $(\sigma_{xx}, \sigma_{xy}) = (0,n) e^2/h$ and $(0, n+1) e^2/h$ with universal critical points $(\frac{1}{2}, n+\frac{1}{2}) e^2/h$, independent of the microscopic details of any models. Such universality at the quantum critical point has also been suggested previously.[5] The universality of the critical exponents appears to have been borne out in all the latest numerical studies with different microscopic models.[2,3,7,8] These works produced values consistent with experimental measurement[9–11] and close to the analytical predicted value[12] $\nu = \frac{7}{3}$ for quantum percolation. The universality of σ_{xx}, however, is still controversial. Extrapolation of dynamical conductivity from Kubo formalisms[8] and from spectra function calculations[7] on short-ranged potentials produced values in reasonable agreement with $\sigma_{xx}^c = \frac{1}{2}$ at the critical point. A recent extensive and direct numerical study[13] on mesoscopic systems with the network model,[2] however, produced $G_c = (0.58 \pm 0.03) e^2/h$, in disagreement with the analytical prediction and previous numerical results. Given the fact that the two-terminal conductance G is not exactly the same as the longitudinal conductivity σ_{xx}, the small but significant difference appears to be resolved[13] in favor of a new universal value for the two-terminal conductance G_c. Experimental attempts[14] to measure the conductance directly have not produced consistent values.

Unlike the Hall conductance, which is exactly quantized as a consequence of topological invariance, the longitudinal conductance itself is a sample-dependent quantity for mesocopic systems with electron coherent lengths exceeding the sample dimension. The universality of the critical conductance in principle applies only to macroscopic systems in which self-averaging is expected. It is not clear at the present what average procedure will produce the experimentally observed quantity. Fluctuation has been the hallmark of quantum transport in mesoscopic systems[15] and its characteristics provide important clues on the system as a whole. It is therefore equally interesting to investigate the probability distribution of the conductance. In particular, it is important to ask whether, under appropriate conditions, the distribution of the conductance $P(G)$ approaches a limiting universal form. In three-dimensional disorder-driven metal-insulator transitions, strong evidence exists on the universality of the conductance distribution in the metallic and insulating phases as well as at the critical point.[16,17] The physical arguments for the existence of such a universality can be equally applied to the localization-delocalization transition in two-dimensional systems.[13,18] Conductance distributions at IQH plateau transitions have been investigated for the network model only.[19] The results compared well with the recent experimental measurement on mesoscopic systems.[20]

In this paper, we investigate the scaling properties of the conductance as well as the critical conductance and its distribution in a two-dimensional system described by a tight-binding model. This is perhaps the simplest system that exhibits the correct critical behavior of the localization-delocalization transition. To our knowledge, no such calculations have been previously reported for the tight-binding model. Our aim is to determine accurately the critical zero-temperature two-terminal conductance in the tight-

binding model and to see whether there is universality, at least between the tight-binding model and the network model investigated by Wang *et al.*[13] Our results are in agreement with the scaling hypothesis and a critical exponent of $\nu = \frac{7}{3}$. More importantly, based on finite size scaling analysis, we obtain $\langle G_c \rangle = 0.506$ at the critical point, very close to the value expected from analytical theory $G_c = \frac{1}{2}$, but in disagreement with the recent extensive numerical study on the network model.[13] This difference, if it persists, has important implications on the universality of the two-terminal critical conductance. We have also analyzed the distribution of the conductance at the critical point. The moments of the distribution are surprisingly similar to that of the network model.

The tight-binding Hamiltonian is given as follows:

$$H = \sum_i \epsilon_i |i\rangle\langle i| + \sum_{\langle ij \rangle} (t_{ij}|i\rangle\langle j| + \text{c.c.}), \qquad (1)$$

where ϵ is a random site energy uniformly distributed within $[-W/2, \ W/2]$. The complex hopping integral t_{ij} carries the phase due to applied magnetic fields via the standard Peierls substitution,

$$t_{ij} = t_0 \exp\left(-\frac{i2\pi e}{hc} \int_i^j \vec{A} \cdot d\vec{l} \right). \qquad (2)$$

The sum is carried over the nearest-neighbor sites $\langle ij \rangle$ only. t_0 is taken as the unit of energy. Periodic boundary conditions are applied in the transverse direction. In continuum, quantized Landau levels under a magnetic field are broadened into Landau bands by impurity scattering. In lattice models, Landau bands form even in ordered systems as a result of degeneracy breaking. These bands will be further broadened by disorder. The lowest Landau band in the tight-binding model has been shown[1] to describe the quantum Hall transition well.

To calculate the zero-temperature two-terminal conductance numerically, we employ the transfer matrix method which obtains the final transmission matrix by multiplications and inversions of transfer matrix. The disordered square sample of size $M \times M$ is sandwiched between two perfect leads of the same width. Both the sample and the leads are governed by the tight-binding Hamiltonian (1). No disorder exists in the leads. The two-terminal conductance is then given by the following multichannel Landauer formula:[21]

$$G = \frac{e^2}{h}\text{Tr}(T^\dagger T), \qquad (3)$$

where T is the total transmission matrix through the disordered sample with the propagating channels in the leads as basis. Keep in mind that G defined here is for one spin only.

For the purpose of investigating the scaling and the critical conductance, we have chosen a fixed magnetic field such that the flux per square is one-eighth of the flux quantum ($f = 1/8$) and a disorder strength $W = 4$. If universality persists, both the critical exponent ν and the critical conductance $\langle G_c \rangle$ are expected not to depend on the applied field and the disorder strength. Based on a previous finite size
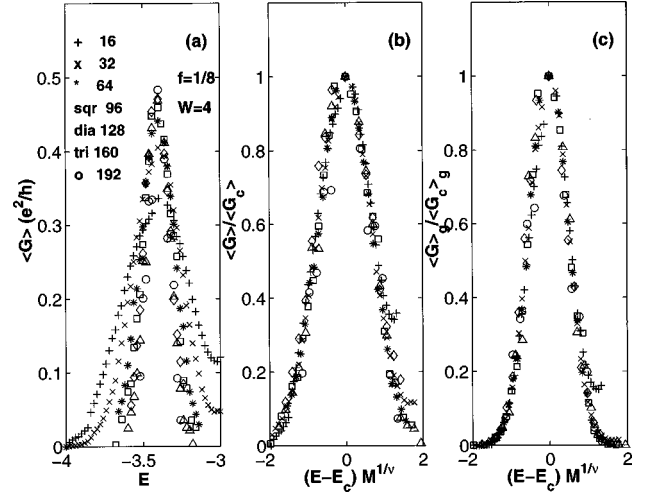

FIG. 1. Average conductance $\langle G \rangle$ in the lowest Landau band. (a) Conductance vs energy for $M = 16, 32, 64, 96, 128, 160$, and 192. Normalized conductance as a function of scaled variable $x = |E - E_c|M^{1/\nu}$ with $E_c = -3.40$ and $\nu = 2.37$ for the arithmetic (b) and the geometric (c) average. The number of samples for each data point ranges from 50 for $M = 192$ and 10 000 for $M = 16$.

scaling study[22] on the localization length, the critical point of the lowest Landau level at this field and disorder strength is accurately known to be at $E = -3.40$. At this disorder, all Landau bands show substantial band mixing except the lowest one. For disordered mesoscopic systems at zero temperature, electrons propagate through the entire sample without being scattered inelastically by phonons. Scattering by random static impurity, on the other hand, produces configuration-dependent conductance fluctuations. We present in Fig. 1(a) the averaged conductance $\langle G \rangle$ in the lowest Landau band for different system sizes. The conductance clearly peaks at the critical energy $E_c = -3.40$ and falls rapidly away from E_c. As the system size increases, the conductance curve become narrower and the peak conductance increases. The continuing narrowing of the width of the conductance curve as system size increases indicates that in the macroscopic limit, only the states at the critical energy can transport electrons across the sample, in agreement with the conventional picture that all states are localized except at the center of the band.[1]

An important property is the scaling of the conductance G as a function of the system size. According to the finite size scaling idea, the conductance is expected to be determined solely by the ratio of the localization length to the system dimension M close to critical point. However, there is known irrelevant finite size corrections such that the scaling is modified as

$$G(E, M) = G(E_c, M)f(\xi(E)/M), \qquad (4)$$

where $\xi(E)$ is the macroscopic localization length at energy E and $f(x)$ is a universal function. The size dependence of the conductance maximum $G(E_c, M)$ represents the irrelevant finite size corrections,

$$G_s(E_c, M) = G_c - aM^{-y_{irr}}. \qquad (5)$$

Utilizing $\xi \sim |E - E_c|^{-\nu}$, we obtain the expression
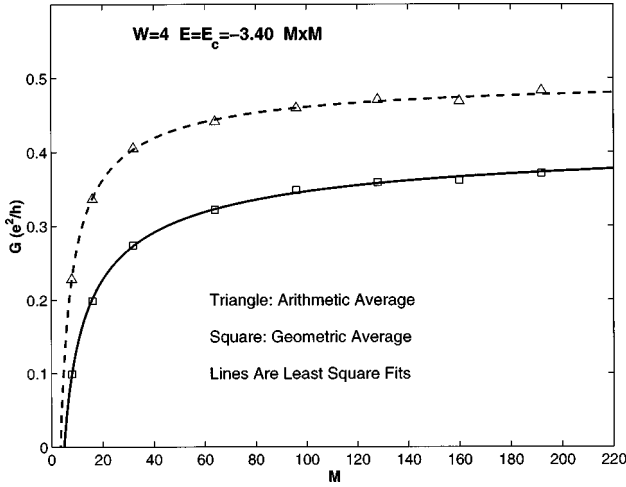
FIG. 2. Conductance at the critical point, $\langle G(E_c,M)\rangle$, as a function of system size M for square samples of $M \times M$. The lines are least square fits to $G_s(E_c,M)=G_c-aM^{-y_{irr}}$. The error bars are smaller than the size of the symbols.

$$G(E,M)=G(E_c,M)f(|E-E_c|^{-\nu}/M)$$

$$=G(E_c,M)F(|E-E_c|M^{1/\nu}).\qquad(6)$$

Should scaling exist in our system as expected, then all of our data for different E and M would collapse on one curve provided the correct values of E_c and ν are chosen. The results of such a scaling procedure are shown in Fig. 1(b) for arithmetic average with $E_c=-3.40$ and the best fit $\nu=2.37$. Scaling behavior is clearly established. Deviation for $M=16$ and for higher energy E is due to the finite size effect and the effect of mixing with higher bands, respectively. Thus scaling of conductance around the critical points indicates a critical exponent consistent with a universal value $\nu=\frac{7}{3}$. The same scaling is obeyed for the geometric average, shown in Fig. 1(c) with exactly the same critical exponent.

Of central importance is the exact value of the conductance at the critical point. As mentioned before, due to the finite size correction from irrelevant fields, the conductance at the critical point depends on system size. In Fig. 2, we present the arithmetic and geometric averaged conductance at the critical point, $E_c=-3.40$. $G(E_c,M)$ increases with increasing size M and will eventually saturate at the critical conductance G_c for macroscopic systems. This is in contrast to the constant amplitude ratio[1] at E_c for the finite size localization length. We mention that in order to achieve good statistics, more than 10 000 samples are taken for M up to 160 and 8 000 samples for $M=192$. To extrapolate to macroscopic systems, we have fitted our data to Eq. (5) with a least square fit (shown as lines in Fig. 3). The most likely fit is determined by minimizing the χ^2 statistics,[23]

$$\chi^2=\sum_i \left[\frac{G(E_c,M_i)-G_s(E_c,M_i)}{\sigma_i}\right]^2,\qquad(7)$$

where the summation i is over all the system sizes and σ_i is the standard deviation of $\langle G(E_c,M)\rangle$. We obtain G_c
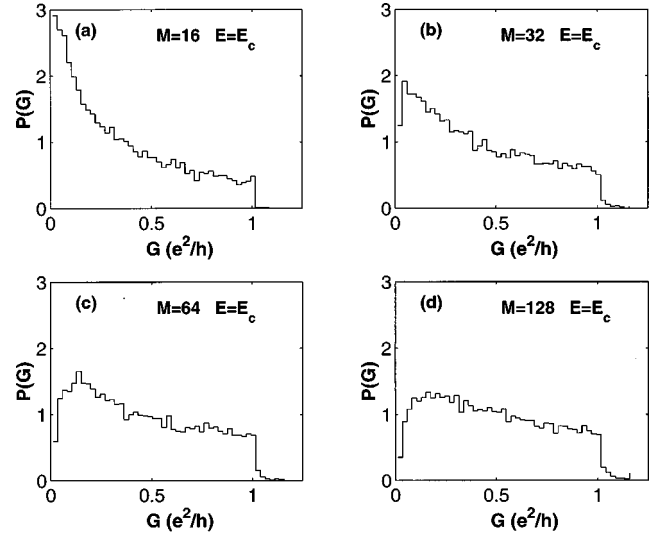

FIG. 3. Distribution of the conductance G at the critical point $E_c=-3.40$ for different sample sizes. (a) $M=16$, (b) $M=32$, (c) $M=64$, and (d) $M=128$. Each size has more than 10 000 samples. Distributions at $M=192$ (not shown here) are almost identical with that of $M=128$ within the statistical fluctuation.

$=0.506$ and $y_{irr}=0.72$ as the best fit with a goodness of fit $Q=0.12$. Fits with Q larger than 0.1 are believable. We have also used the projection method[23] to estimate the confidence limit. Brackets of $[0.499,0.511]$ and $[0.495,0.517]$ are obtained with a confidence level of 95.4% and 99.73%, respectively. Our results agree with the analytical assertion that the universal longitudinal conductance G_c is $\frac{1}{2}$. The difference between our results for the tight-binding model and the equally extensive results for the network model,[13] if it persists, could indicate that the critical two-terminal conductance is not universal. The extrapolated value for the geometric average is $\langle G_c\rangle_g=0.438$ with $y_{irr}=0.52$ and a goodness of fit $Q=0.50$. An interesting remark is that from the reported value of higher moments of G, we infer that in the network model, the geometrically averaged conductance $\langle G\rangle_g \approx 0.5$.

The distribution of the conductance also shows interesting properties. At the critical point, this distribution is broad and ranges between 0 and 1. Fluctuations, as measured by the standard deviation, are of the same order of magnitude as the average conductance itself. For localized state, the distribution is known to be Poissonlike. At critical points, it has been proposed that the conductance distribution should be universal independent of the size of the system. This assertion is based on the fact that there is no length scale since the localization length diverges at the critical point. However, we know there is non-negligible finite size corrections to the scaling, as shown in the analysis of the G_c. Thus, the conductance distribution at the critical point does show size dependences [Figs. 3(a)–3(d)]. The probability distribution is broad, ranging from 0 to 1. It also shows progressive development of a dip around $G=0$ as the whole distribution flattens. For two-dimensional (2D) systems, analytical descriptions of the statistical distribution of the conductance are lacking. Eventually for very large sizes, the distribution satu-

rates to the final, presumably universal, distribution for macroscopic systems. Calculation of higher moment of the conductance, $\langle G^n \rangle$, results in values 0.08, 0.013, 0.003, and 0.001 for $n=2$, 4, 6, and 8, respectively. These values closely resemble the results reported by Wang *et al.*[13] The probability distribution at the 2D quantum critical point is quite different from that of the 3D systems[16,17]. We also point out that the distribution in ln$G$, although not Gaussian,

has much better central tendency. The universality of these distributions will be examined in future work.

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