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Quantum oscillations in one-dimensional metal rings: Average over disorder

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We study the Aharonov-Bohm effect in single normal-metal rings and show that averaging the transmission coefficient T over disorder gives oscillations with a period of a half-flux quamtum. As the elastic scattering gets stronger, the periodicity of oscillation of the conductance, which is related to T, gradually changes to a full-flux quantum, in agreement with recent experiments.

There have been intense theoretical and experimental efforts in recent years to understand the behavior of conductance of normal-metal conductors at low temperatures. Magnetoresistance oscillations of Aharonov-Bohm effect type in disordered rings and cylinders with flux period $hc/2e$ were first predicted by Al'tshuler, Aronov, and Spivak (AAS),[1] and have been verified experimentally for disordered cylinders by several groups.[2-5] Experiments on small rings, on the other hand, showed complicated features. Until recently, no periodic oscillations had been clearly observed in single-metal rings. However, Webb, Washburn, Umbach, and Laibowitz[6] have observed signals of oscillations with period hc/e in small gold rings, and very recently Chandrasekhar, Rooks, Wind, and Prober[7] have unambiguously observed $hc/2e$ oscillations at low magnetic fields and weaker hc/e oscillations at higher magnetic fields, on single aluminum and silver rings.

The AAS theory, which predicts $hc/2e$ oscillations that decay rapidly at high fields, is based on the weak-localization theory of the ensemble-averaged magnetoresistance.[8] Carini, Muttilab, and Nagel[9] have also predicted $hc/2e$ oscillations and argued that the origin of these oscillations of the conductance could be traced to the existence of degeneracies and time-reversal invariance of the Hamiltonian after ensemble averaging. An alternative approach,[10-12] based on calculating the transmission coefficient, gives a fundamental period of hc/e in rings at zero temperature. Although higher harmonics do exist, they become dominant only at special conditions and are *not* equivalent to the effect predicted by AAS.[1] No ensemble average is taken in this theory.[10-12] It was suggested[12,13] that $hc/2e$ contribution could become dominant in the multichannel case due to the random contribution associated with flux-independent phases. In addition, in very small samples there is the extra complication of aperiodic fluctuations added to the magnetoresistance.[14]

In the present paper, we will first give an explicit formula of the transmission coefficient, which is related to the conductance by Landauer's formula,[11,12] for a symmetric normal-metal ring. Then we will show that in the weak-scattering limit the conductance is a periodic function of the flux through the hole in the conductor with the $hc/2e$ period after averaging over the phases of the scatterers. We will also show that increasing scattering will destroy the $hc/2e$ oscillations, and the period will change to hc/e, in agreement with the latest experiments.[7]

Following Büttiker, Imry, and Landauer,[11] we describe a metal ring by two effective parallel elastic scatterers with two leads. The leads can be described by an S matrix which relates the amplitudes of the three incoming waves to the three amplitudes of the outgoing waves. The matrix S has to be unitary and symmetric due to the physical requirement of probability conservation and time-reversal invariance. A simple choice that S is real and symmetric with respect to the two branches of the circle is given by[11]

$$S=\begin{pmatrix} -(a+b) & \epsilon^{1/2} & \epsilon^{1/2} \\ \epsilon^{1/2} & a & b \\ \epsilon^{1/2} & b & a \end{pmatrix}, \tag{1}$$

where $a=[(1-2\epsilon)^{1/2}-1]/2$, $b=[(1-2\epsilon)^{1/2}+1]/2$, and ϵ, $0\leq\epsilon\leq\frac{1}{2}$, is called the coupling constant, since $\epsilon=0$ and $\epsilon=\frac{1}{2}$ correspond to decoupling and strong coupling of a ring with leads, respectively. Scatterers are described by a transfer or $\underline{t}$ matrix. Since we consider a one-dimensional system, the $\underline{t}$ matrix is given by

$$\underline{t}=\begin{pmatrix} 1/t^* & -r^*/t^* \\ -r/t & 1/t \end{pmatrix}. \tag{2}$$

Here $t=T_s^{1/2}e^{i\phi}$ is the transmission amplitude of the scatter, T_s the transmission probability, and ϕ the phase change in the transmitted wave. The reflection amplitude of the scatter is given by $r=R_s^{1/2}e^{i\phi}e^{-i\pi/2}$. Generally, t and r are functions of electron energy, magnetic field, and disorder of the metal. Following the formalism developed by Büttiker *et al.*,[11] the total transmission coefficient is obtained as a function of magnetic flux Φ, T_s, and phases ϕ_1 and ϕ_2 for the two branches of the ring. We find that the total transmission coefficient T is given by $T=|a_2'|^2$, where a_2' is

$$a_2'=\frac{2i\epsilon\sqrt{T_s}\,[(\sin\phi_1+\sin\phi_2+2\sqrt{R_s})\cos\pi(\Phi/\Phi_0)+i(\sin\phi_1-\sin\phi_2)\sin\pi(\Phi/\Phi_0)]}{2b^2T_s\cos2\pi(\Phi/\Phi_0)-(1-2\epsilon)\exp[i(\phi_1+\phi_2)]-\exp[-i(\phi_1+\phi_2)]+2a^2\cos(\phi_1-\phi_2)-(M)}, \tag{3}$$

where

$$M=+2R_s(b^2-2a^2)+2ia\sqrt{R_s}\{(1-2\epsilon)^{1/2}[\exp(i\phi_1)+\exp(i\phi_2)]+[\exp(-i\phi_1)+\exp(-i\phi_2)]\}.$$

Φ is the magnetic flux through the hole and $\Phi_0=hc/e$ is the flux quantum. We immediately see from Eq. (3) that the transmission coefficient T is a periodic function of flux with a period of the flux quantum hc/e. These results are in agreement with those of Refs. 10 and 12. Equation (3) can be simplified a lot if we approach the weak-scattering limit where $T_s=1$ for the two scatterers in the branches of the ring, while the phase changes in the transmitted waves in the two scatterers are ϕ_1 and ϕ_2, respectively. In this limit, the total transmission coefficient is given by

$$T=|a_2'|^2=\frac{\epsilon^2[\sin^2\phi_1+\sin^2\phi_2+2\sin\phi_1\sin\phi_2\cos 2\pi(\Phi/\Phi_0)]}{[b^2\cos^2\pi(\Phi/\Phi_0)-(1-\epsilon)\cos(\phi_1+\phi_2)+a^2\cos(\phi_1-\phi_2)]^2+\epsilon^2\sin^2(\phi_1+\phi_2)} \quad . \tag{4}$$

For $\phi_1=\phi_2$, Eq. (4) agrees with the Eq. (4.25) of Ref. 11, which has been studied very carefully and always shows a periodicity of full flux. We see, from Eq. (4), that for a single configuration of the disordered ring in the presence of magnetic field, the transmission coefficient T will always be a periodic function of full flux at zero temperature. However, if we average over the disorder equation (4) we might obtain the half-flux oscillation as was speculated.[11,13] Note that in the weak-scattering limit $T_s=1$ only ϕ_1 and ϕ_2 are randomly distributed with a rectangular probability distribution between 0 and 2π. In the weak-scattering limit, this is shown to be true[15] for the one-dimensional Anderson model with diagonal disorder. Therefore, in this regime, averages over the phases ϕ_1 and ϕ_2 must be taken, and the macroscopic properties of the sample are the ensemble-averaged quantities. We use uniform phase distribution. Hence, the geometric average of the total transmission coefficient is given by

$$\langle T\rangle_g=e^{\langle \ln T\rangle} \quad , \tag{5a}$$

where

$$\langle \ln T\rangle=\frac{1}{(2\pi)^2}\int_0^{2\pi}d\phi_1\int_0^{2\pi}d\phi_2\ln T(T_s,\phi_1,\phi_2,\epsilon,\Phi) \quad . \tag{5b}$$

We also calculate the arithmetic average of T which is a direct average of T in Eq. (5b). T, in Eq. (5b), can be taken either from Eq. (4) or Eq. (3). For $T_S=1$, Eq. (5b), together with Eq. (4), indeed shows that such an average over disorder will change the period to $hc/2e$. This can be seen clearly from the relation

$$T(T_S=1,\phi_1,\phi_2,\epsilon,\Phi+\Phi_0/2)=T(T_S=1,\phi_1+\pi,\phi_2,\epsilon,\Phi) \quad , \tag{6a}$$

hence,

$$\langle T\rangle_{\Phi+\Phi_0/2}=\langle T\rangle|_{\Phi} \quad . \tag{6b}$$

This is correct for the geometric as well as the arithmetic average of the transmission coefficient T for $T_S=1$. The geometric average $\langle T\rangle_g$ of T for different coupling constants ϵ is plotted in Fig. 1. The striking feature is that $\langle T\rangle_g/\epsilon^2$ is extremely insensitive to changes of the coupling constant. From Fig. 1 we clearly see that T is a periodic function of half-flux quantum $\Phi_0/2$. As a comparison, we plot in Fig. 2 the arithmetic average $\langle T\rangle_a$ of the total transmission coefficient T as a function of the flux through the hole of the ring. In this case too, the period of oscillation of $\langle T\rangle_a$ is half-flux quantum, but $\langle T\rangle_a/\epsilon^2$ does depend on the coupling constant ϵ.

Similar oscillations have been seen in the work of Carini *et al.*[9] for the participation ratio. Our results suggest that the ensemble-average picture presented in Ref. 9 and the perturbation theory of AAS[1] and Bergman[8] is not so different from that expressed in the transfer matrix picture,

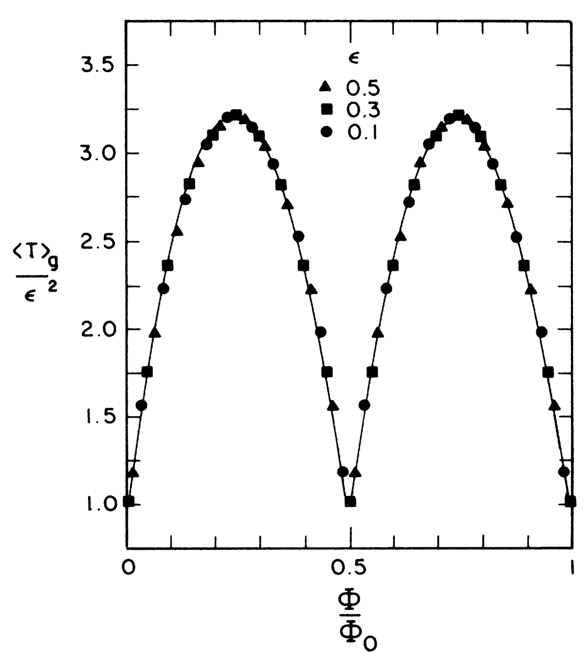


FIG. 1. $\Phi_0/2$ oscillations of the geometric average of the total transmission coefficient for different degrees of coupling for weak scattering $T_S=1$.

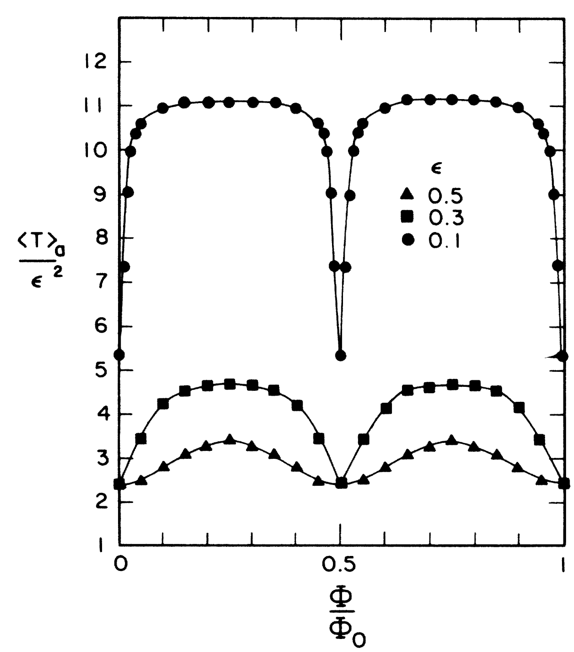
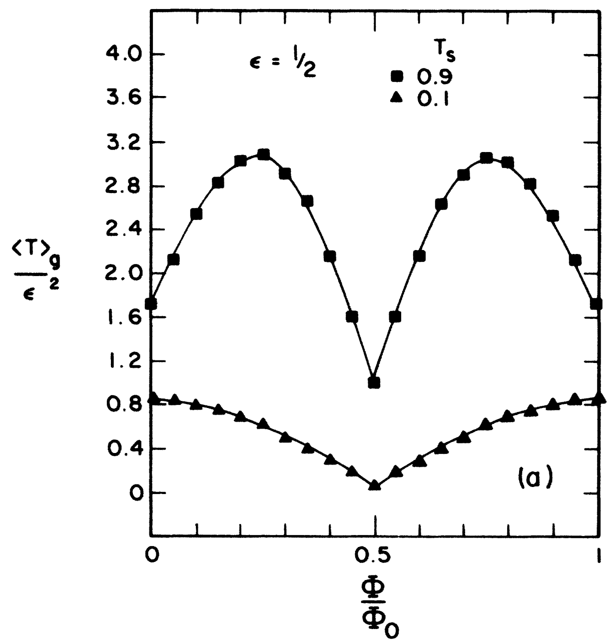


FIG. 2. $\Phi_0/2$ oscillations of the arithmetic average of the total transmission coefficient for different degrees of coupling for weak scattering $T_S=1$.

provided an ensemble average over disorder is taken. It is very interesting that if we average total transmission coefficient T over the phases ϕ_1 and ϕ_2 [Eq. (3)] when $T_s < 1$, we find that $\langle T \rangle_g$ and $\langle T \rangle_a$ are not periodic functions of half flux anymore, but of full flux. This is clearly shown in Figs. 3(a) and 3(b), where $\langle T \rangle_g / \epsilon^2$ and $\langle T \rangle_a / \epsilon^2$ are, respectively, plotted as a function of magnetic flux through the hole of the ring for $\epsilon = \frac{1}{2}$ and different values of T_s.

By carefully examining Figs. 3(a) and 3(b), we notice that the transmission coefficient has a full-flux oscillation, but there is also an appreciable component of a half flux. The oscillations of the transmission coefficient in Fig. 3(b) are very similar to those seen in Fig. 1(c) of Ref. 9 for small size rings, for the participation ratio. They interpreted[9] their numerical results as half-flux oscillations because their time-reversal-symmetry arguments will persist for any size system. Most astonishing is our result that even after ensemble averaging for $T_s < 1$, the transmission coefficient is periodic in hc/e. Although the results are not periodic with period $hc/2e$, there is also a significant decrease in the transmission coefficient for half-integer values of Φ/Φ_0 [see Fig. 3(b)]. The general philosophy in this field is that ensemble averaging kills the hc/e. We believe that this statement might indeed lack precision; presumably, it depends on the ensemble average being considered. Expressed otherwise, ensembles which are not "wild" enough might not be sufficient to lead to the self-averaging of the hc/e component. The important question then is what is a physically relevant ensemble, i.e., which ensemble incorporates the variations from member to member which we would expect in a real system? In this work as an ensemble average we take that one, in which the distribution of phases is uniform. We know that this is true for the weak-scattering limit,[15] but it is possible that the distribution is no longer uniform for strong disorder, and this might be the reason for differences between our results and those of Ref. 9. Note that as T_s decreases from the value one, which corresponds to the weak-scattering limit, the period of oscillation of $\langle T \rangle_g$ and $\langle T \rangle_a$ gradually shifts towards the full-flux quantum. So for the strong-scattering case, i.e., $T_s \ll 1$, the hc/e period would become dominant. This is simply related to the fact that the $hc/2e$ period oscillation involves backscattering interference[8] in which electrons effectively circle around the ring twice. Therefore, for the strong-scattering case, electron waves would be greatly attenuated and phase coherence around the whole ring would almost be lost, and the contribution with the hc/e period would be observed. It can be argued that as the magnetic field is increased, the transmission amplitude T_s of the scatterer will be decreased, and therefore $\langle T \rangle_g$ or $\langle T \rangle_a$ will show full-flux quantum oscillations in agreement with experiments[7] which study the magnetic field dependence of T. Very recently, Stone and Imry[16] have argued that increasing temperature will cause single-ring self-averaging; the flux periodicity of the magnetoresistance oscillations becomes $hc/2e$. Of course, at zero temperature with no self-average, the oscillations are of the hc/e type.

To summarize, we have shown that the transmission coefficient of a normal-metal ring with contacts will oscillate as a function of the magnetic flux with a period of half-flux quantum in the weak-scattering case. For the strong-scattering case full-flux quantum oscillations are dominant. All of these results are correct for zero temperature. To make a comparison with the experimental results[6,7] we have to define the important characteristic lengths and discuss their dependence on temperature, disorder, and magnetic field. One is the electron-phase coherent length l, which is the distance that an electron travels before randomly changing its wave-function phase. l is, roughly speaking, the mean free path which for a typical metal[7] is of the order of 10–100 Å and independent of temperature. The inelastic diffusion length is $l_{in} = (D\tau_{in})^{1/2}$, where D is the diffusion constant (assumed to be temperature independent) and τ_{in}



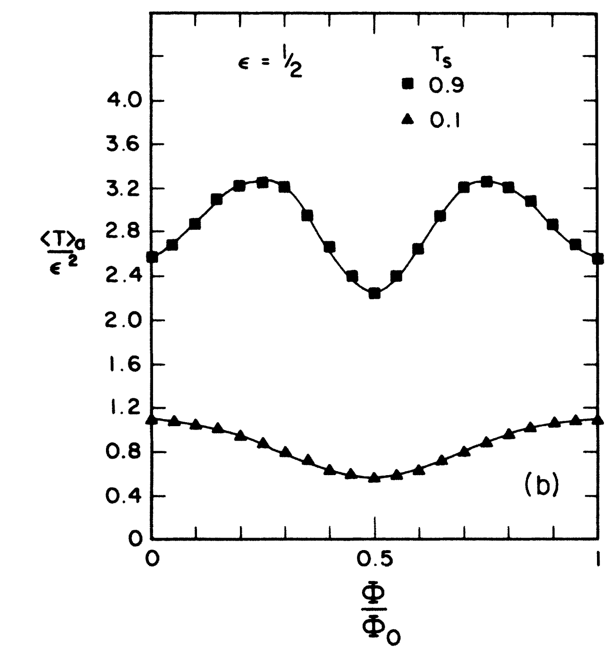


FIG. 3. Average of the total transmission coefficient as a function of magnetic flux through a hole for different T_S for $\epsilon = \frac{1}{2}$. (a) Geometric average $\langle T \rangle_g$. (b) Arithmetic average $\langle T \rangle_a$.

is the mean time between inelastic collisions. It is expected that τ_{in} is inversely proportional to the temperature. l_{in} can be larger than 1 μm at low temperatures (1 $\mu m = 10^4$ Å).[7] The localization length l_c has to be of the order of l_{in} if one wants to see these electron interference effects. Finally, the magnetic length is $l_H \sim (hc/eWH)$, where H is the applied magnetic field and W the width of the sample. The ratios of these lengths to the sample length L governs the size (or the presence) of the oscillations. The condition for observing the Aharonov-Bohm effect with half-flux quantum in disordered rings is that $l_{in} \sim l_c \sim l_H \gtrsim L >> l$, where L is the perimeter of the ring. This has to be distinguished from Aharonov-Bohm resistance oscillations in very pure single crystal with a long mean free path $l >> L$ and with a period of hc/e. In the disordered ring, the oscillations with half flux will gradually give way to hc/e oscillations as we increase either the magnetic field or the disorder. In these two cases l_c or l_H will decrease and phase coherence around the whole perimeter of ring L will be destroyed, including the $hc/2e$ oscillation. This picture agrees with the experimental results.[7] Finally, by increasing the temperature, l_{in} decreases and eventually will become smaller than $L/2$. This phase incoherence introduced by increasing the temperature will destroy both the periods of oscillations of the magnetoresistance.

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