Static Conductance and Scaling Theory of Localization in One Dimension

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It is shown that the Ohmic dc conductance of a one-dimensional system equals to either $G_p = (e^2/\pi\hbar)T$ or $G_c = (e^2/\pi\hbar)T/(1-T)$, depending on whether the system is connected to a perfect one-dimensional conductor or to a classical wire (behaving as a current source), respectively. The parameter $\beta \equiv d\langle \ln g' \rangle / d \ln L$, where L is the length of the system, depends only on $\langle \ln g' \rangle$, so that $\beta = \langle \ln g' \rangle$, with $g' = (\pi\hbar/e^2)G_p$.

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A very significant advance in producing a scaling theory of electronic localization in disordered systems was made by Thouless and co-workers[1,2] and by Abrahams *et al.*[3] In particular, Thouless and co-workers[1,2] have suggested that the sensitivity to boundary conditions of the energy levels can be closely related to the conductance G of the system at that energy. Thus it is the conductance that determines whether or not the states are localized. Abrahams *et al.*[3] pursued the ideas of Thouless[1] to propose a scaling theory of localization with only one parameter. They chose the dimensionless conductance $g = G/(e^2/2\hbar)$ of small sample whose sides are of length L as the scaling parameter and they examined how g changes with the sample size L. They assumed that the quantity $\beta \equiv d \ln g / d \ln L$ is a monotonic and nonsingular function of g only and they were able to conclude that $g \to 0$ as $L \to \infty$ for any disordered system of dimensionality lower than or equal to 2. In particular, for two dimensions the conductance was found to cross over smoothly from a logarithmic to an exponential decrease with L.

The purpose of this Letter is to evaluate the zero-temperature dc conductance G of a one-dimensional (1D) disordered system of length L and thus check the scaling theory of localization. Our numerical results show that in 1D, β is in fact a nondecreasing function of $\langle \ln g \rangle$ only.

The electronic states of disordered 1D lattices have been extensively studied. In contrast, less effort has been devoted to understanding transport properties. Here we present a calculation for the $T = 0$ dc conductance of a *finite* system. Consider a system of length L connected on both sides to a matching "wire" of total length L' with $L' \to \infty$. As a matching "wire" we define one which maximizes the geometric mean of the transmission coefficient of the resulting arrangement. The dc conductance of the system, G, is defined as I/V in the limit $L' \to \infty$ first and then $\omega \to 0$. I is the current through our system, $V = EL$ is its voltage, and E is the electric field which is taken as uniform within our system. Using linear-response theory,[4] we obtain for the real part of the current $I(x)$ of the system,

$$I(x) = (\pi e^2\hbar/2\omega m^2) \sum_{\alpha,\beta} W_{\alpha\beta}(x) \int W_{\alpha\beta}^*(x') E(x') dx' [\delta(\omega - \omega_{\beta\alpha}) - \delta(\omega + \omega_{\beta\alpha})], \tag{1}$$

where

$$W_{\alpha\beta}(x) = \psi_\alpha^* \partial\psi_\beta/\partial x - \psi_\beta \partial\psi_\alpha^*/\partial x, \tag{2}$$

$\omega_{\beta\alpha} = (E_\beta - E_\alpha)/\hbar$, and m and e are the mass and the charge of the electron, respectively. The integral extends over the whole length $L' + L$ to cover all the cases that we will examine. The summation over the index α (β) includes all eigenstates ψ_α (ψ_β) with $E_\alpha < E_F$ ($E_\beta > E_F$), respectively, and E_F is the Fermi energy; the spin summation has already been performed in Eq. (1). In what follows the transmission coefficient T' of the "wire" is taken as $T' = 1$.

Near the left and right ends of the "wire" connected to our system the normalized eigenfunctions can be written as $\psi_L = A e^{ikx} + B e^{-ikx}$ and $\psi_R = F e^{ikx} + G e^{-ikx}$, respectively. A, B are related to F, G through a transfer matrix involving the transmission and reflection amplitudes t and r ($|t|^2 = 1 - |r|^2 = T$ is the transmission coefficient) of our system: With use of unitarity and time-reversal symmetry, the matrix elements of the transfer matrix are $M_{11} = 1/t$, $M_{22} = 1/t^*$, and $M_{12} = -r/t = -M_{21}$, and phase of t = phase of $r + \frac{1}{2}\pi$. Similarly F, G are related to A, B through the t', r' of the "wire." Using these relations and putting $|t'| = 1$, one can show that $A = 1/(2L')^{1/2}$ and $B_\pm = (r \pm t)/(2L')^{1/2}$. In the limit $\hbar\omega \ll E_F$ one then obtains for $x\omega_{\beta\alpha}/v_F \ll 1$ ($x = 0$ is the center of our system),

$$|W_{\alpha\beta}(x)| = (2k_F/L')\sqrt{T}, \tag{3}$$

where one of the eigenstates ψ_α, ψ_β is associated with $B_+ \sim (r+t)$ and the other with $B_- \sim (r-t)$; otherwise $|W_{\alpha\beta}| = 0$.

Case 1.—In this case we choose the external "wire" to be a perfect 1D conductor, with no field applied across it. Substituting Eq. (3) in Eq. (1) and performing the summation over the eigenstates and the integration over the length L of the system, we obtain for the Ohmic dc conductance G_p

$$G_p = (e^2/\pi\hbar)T. \tag{4}$$

It should be noted that Abrikosov and Ryzhkin,[5] who considered the particular case of a white-noise randomness, obtained an expression for G which, in the limit $T \to 1$, coincides with Eq. (4). Furthermore, the exact results of Albers and Gubernatis[6] for the ac conductivity of a perfect periodic system averaged over a frequency range at least equal to the level spacing satisfied the inequality $G < e^2/\pi\hbar$. However, Landauer[7] and recently Anderson *et al.*[8] and Azbel[9] have found that the conductance equals to G_c, where

$$G_c = (e^2/\hbar\pi)T/(1-T). \tag{5}$$

The difference between Eqs. (4) and (5) is due to the fact that in deriving Eq. (5) the system has not been connected to a perfect conductor as in our case, but to a current source. Then, as we show below in case 2, Landauer's result is obtained.

Case 2.—Here the "wire" is chosen to be equivalent to a current source. This is achieved by considering the "wire" to be a classical conductance G' (i.e., one with a local current-field relation implying that $G' \sim 1/L' \to 0$) across which a voltage $V' = E'L' \to \infty$ is applied so that $G'V' = I$ is the constant current. Such a classical behavior requires[10] that the eigenfunctions in the "wire" have constant amplitude (which implies $T' = 1$) and random phases so that[10]

$$W_{\alpha\beta}(x)\int W_{\alpha\beta}^*(x')dx' = \gamma|W_{\alpha\beta}(x)|^2 l, \tag{6}$$

where l is the mean free path[5,11] and γ is a constant of order unity. Equation (6), which is due to cancellations arising because of the random phases, is crucial in producing a local current-field relation. It will be argued below that a strictly 1D wire cannot exhibit this classical behavior. Nevertheless, if we use Eq. (6), together with Eqs. (3) and (1), and perform the summation over the eigenstates, we obtain for the current in our system,

$$I = (e^2\pi\hbar)T(\gamma l/L'V' + V). \tag{7}$$

In the limiting case $L = V = 0$, where $T = 1$, we obtain from Eq. (7) that $G' = (e^2/\hbar\pi)(\gamma l/L')$, which for $\gamma = 1$ is just the classical result. Combining this expression for G', Eq. (7), and $I = G'V' = GV$, we arrive at Eq. (5) independently of the exact value of γ.

For strictly 1D systems the Wronskian $W_{\alpha\beta}(x)$ is constant both in phase and amplitude over distances much less than v_F/ω. As a result the current-field relation is not local, which accounts both for the nonadditivity of the resistances[8] as well as the dependence of the total conductance on the external "wire." Thus the hypothesis on which Eq. (5) is based, i.e., the existence of a classical 1D conductance, is not rigorously valid. One may avoid this difficulty by considering a real 3D wire which can be made to behave classically. However, in this case, one has to face questions associated with the connection of the external 3D wire to our 1D system. These points illustrate the experimental difficulties one may have in realizing the conditions under which Eq. (5) is valid. Similarly, although there is no logical inconsistency in the derivation of Eq. (4), realizing experimentally a long, perfect 1D conductor (i.e., one with $l \gg L' \gg v_F/\omega$) is not easy. Hence both Eqs. (4) and (5) involve a certain idealization which makes their connection to experimental situations unclear. The regime $T \ll 1$, where both Eqs. (4) and (5) produce essentially identical results, may be easier to realize experimentally. In the $T \to 1$ regime, the experimental results may be substantially different from either or both Eqs. (4) and (5), depending on the particular details. The difference between Eqs. (4) and (5) in the $T \to 1$ regime is not surprising, since case 1 allows acceleration of the electrons (which implies $\mathrm{Im}G_p = ne^2/Lm\omega \to \infty$ as $\omega \to 0$), while case 2 allows a steady incoherent beam of electrons to pass through the system without any voltage (which implies $\mathrm{Re}G_c \to \infty$). G_c gives the classical result for $L/l \ll 1$ partly because, then, the amplitude of the wave function is constant over the length of the system and partly because the external classical wire provides the phase randomization which the 1D system cannot achieve by itself. The case-1 configuration has no phase-randomizing mechanism and, as a result, never behaves as a classical wire; for the same reason the current in the perfect conductor varies with distance as $\cos(x\omega/v_F)$.

Eqs. (4) or (5) (Ref. 12) are very convenient for a numerical study since the transmission coefficient for a 1D disordered system can be easily

obtained. We study here the 1D Anderson model described by a Hamiltonian of the form $\langle l|H|m\rangle = \epsilon_l \delta_{lm} + V(\delta_{m,l+1} + \delta_{m,l-1})$, where $|l\rangle$ is an atomic-like orbital centered around the lattice point l, ϵ_l is distributed randomly between $-\alpha$ and α for $1 \leq l \leq N+1$ and is equal to zero otherwise, and V is a positive constant. The amplitudes c_{N+2} and c_{N+3} of the eigenfunction with eigenenergy E can be taken as $c_{N+2} = 1$ and $c_{N+3} = e^{ika}$, where $E = 2V \times \cos ka$, and a is the lattice constant. Then one can calculate, through the recursion relations $\epsilon_n c_n + V c_{n+1} + V c_{n-1} = E c_n$, the amplitudes c_0 and c_1 and thus obtain the transmission coefficient T by the relation

$$T = |\exp(2ika) - 1|^2 / |c_1 - c_0 \exp(-ika)|^2.$$

Our numerical results show that the random variables $T(L)$ (where $L = Na$) and $1/T(L)$ are not normally distributed.[12,8] In fact, their probability distributions have very long tails which completely dominate the behavior of their means. Furthermore, the ratio of their standard deviations σ to their means is an increasing function of L/l_c, where l_c is the localization length of the system.[13] The random variable $\ln T$ has a smoother distribution[8] but even that is nonnormal for $L \lesssim l_c$. On the other hand, $\ln T$ is normally distributed[14] for $L \gg l_c$ with $\sigma_{\ln T}/|\langle \ln T\rangle| = (l_c/L)^{1/2}$. As shown in Fig. 1, the arithmetic, geometric, and harmonic mean values of T are very different, especially for large values of L/l_c. Note that $\langle \ln T\rangle$ is within numerical uncertainties equal to $-2L/l_c$. From Fig. 1 we see that both the harmonic (which equals to the inverse of the average resistance if $\mathrm{Im}G_p = 0$) and especially the arithmetic mean, in contrast to the geometric mean, do not seem to be functions either of the ratio L/l_c only or the ratio L/l only.

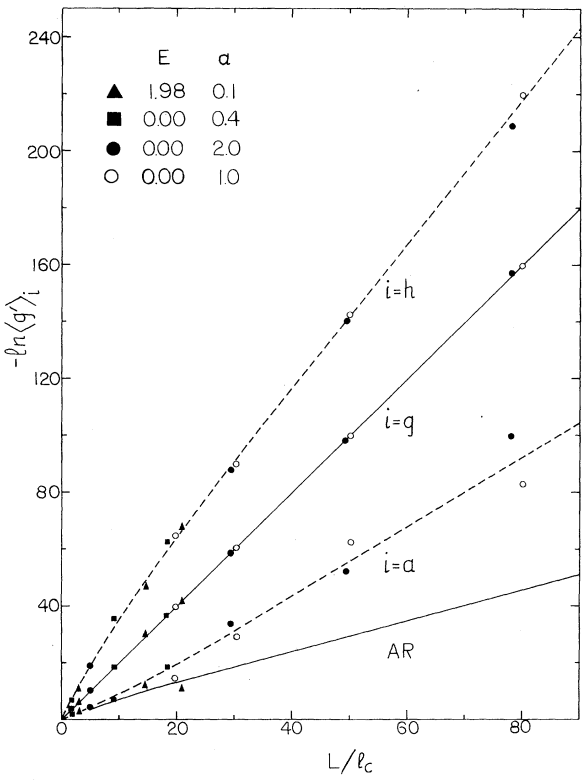

FIG. 1. Various means of the normalized Ohmic conductance $g' \equiv (\pi\hbar/e^2)G_p = T$ vs L/l_c (L is the length of the specimen and l_c is the localization length) for different values of the energy E and the degree of disorder α. The geometric mean $\langle g'\rangle_{\mathrm{geom}}$ follows the straight line denoted by $i = g$. The arithmetic and the harmonic means are clustered around the dashed lines (guides to the eye) denoted by $i = a$ and $i = h$, respectively. The line denoted by AR is the prediction of Ref. 5 for the normalized conductance.

Finally, our explicit results allow us to check the basic assumption of the renormalization approach[1-3] to the localization problem, namely that of a single-parameter scaling equation of the form $\beta \equiv d\ln g/d\ln L = f(g)$; we found indeed that this assumption is correct, provided that a proper choice for g is made. The choice for g which produces the simplest function $f(g)$ is $g = \langle g'\rangle_{\mathrm{geom}}$, where $\langle\ \rangle_{\mathrm{geom}}$ denotes geometric mean and $g' \equiv (\pi\hbar/e^2)G_p = T$. Then

$$\beta = \ln g = \langle \ln g'\rangle, \tag{8}$$

as shown in Fig. 2. Note that β intercepts the $\langle \ln g'\rangle$ axis at zero and cannot be defined for high-

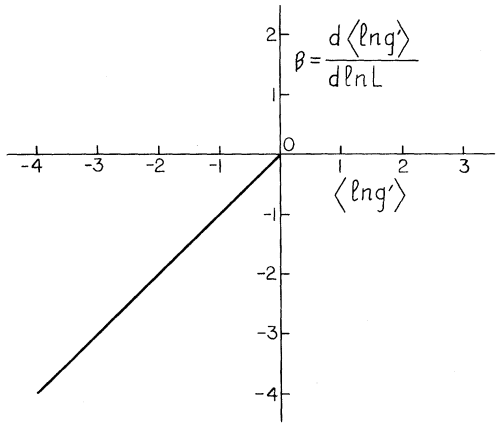

FIG. 2. The scaling function β vs $\langle \ln g'\rangle$ for a tight-binding 1D disordered system (see text). The β vs $\langle \ln g'\rangle$ line terminates at the origin.

er conductance since such a conductance cannot be realized according to Eq. (4). If one takes G_c as the conductance, the natural choice for g is $g = (\pi\hbar/e^2)\langle G_c\rangle_{\text{geom}} = \langle T/(1-T)\rangle_{\text{geom}}$. The resulting β again seems to be a function of g only; for small g, β behaves as in Eq. (8) while for large g, β can be expanded as $\beta = -1 - b_1/g + b_2/g^2 + \cdots$; from our numerical work we found that $b_1 = 3.5 \pm 0.5$ and $b_2 = 9 \pm 2$. In Ref. 8, g has been chosen as $g = \langle T\rangle_{\text{geom}}/(1-\langle T\rangle_{\text{geom}}) = \langle g'\rangle_{\text{geom}}/(1-\langle g'\rangle_{\text{geom}})$; in this case, for large g, we found, using our numerical results, that $\beta = -1 - 1/2g + 1/6g^2 + \cdots$, in agreement with Ref. 8.

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